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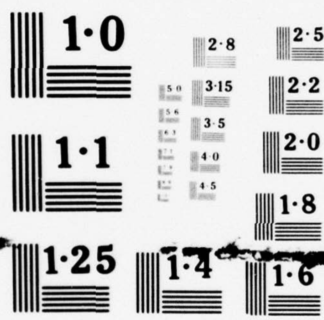

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DYNAMIC STABILITY OF ONE-DIMENSIONAL NONLINEARLY VISCOELASTIC BODIES

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ABSTRACT

Systems of partial differential equations governing the motion of one-dimensional bodies subject to internal friction are treated. The implicit function theorem is used to linearize the equations about an equilibrium solution, and criteria are developed for the stability of the equilibrium solution.

AMS (MOS) Subject Classifications: 35B40, 35K45, 73-35
Key Words: Dynamic stability, Rod theories, Viscoelasticity
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EXPLANATION

This paper treats the general equations of rod theory; it does not deal with any specific application. The assumption of internal friction is introduced both as a better approximation to reality and to obtain tractable (parabolic rather than hyperbolic) differential equations. Under this assumption, we show that the desirable theorems for the nonlinear equations: existence, uniqueness, and stability, follow from the analogous theorems for the linearized equations.

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DYNAMIC STABILITY OF ONE-DIMENSIONAL NONLINEARLY VISCOELASTIC BODIES

Russell C. Browne

1.0. Introduction.

In this paper we consider the problems of existence, uniqueness, and stability for the quasilinear partial differential equations governing the motion of nonlinearly viscoelastic one dimensional bodies. These equations have the form

$$(1.1)\qquad \mathbf{A}(\mathbf{u},s)\cdot\mathbf{u}_{tt} + \mathbf{a}(\mathbf{u},\mathbf{u}_t,s) - \frac{\partial}{\partial s}\mathbf{m}(\mathbf{u}_s,\mathbf{u},\mathbf{u}_{st},\mathbf{u}_t,s)$$
$$+ \mathbf{n}(\mathbf{u}_s,\mathbf{u},\mathbf{u}_{st},\mathbf{u}_t,s) = \mathbf{f}(\mathbf{u}_s,\mathbf{u},\mathbf{u}_t,\mathbf{s},t)$$
$$\text{for}\quad s_1 < s < s_2 \quad\text{and}\quad t > 0 .$$

In equation (1.1), $\mathbf{u}$ is a function of s and t with values in $\mathbb{R}^N$, $\mathbf{a}$, $\mathbf{m}$, $\mathbf{n}$, and $\mathbf{f}$ are functions of the indicated arguments with values in $\mathbb{R}^N$, and $\mathbf{A}$ is a function with values in $L(\mathbb{R}^N;\mathbb{R}^N)$, the space of linear transformations on $\mathbb{R}^N$.

Boundary conditions for equation (1.1) may be stated parametrically as

$$(1.2)\qquad \mathbf{u}(s_\alpha,t) = \mathbf{g}_\alpha(\mathbf{v}_\alpha,t), \qquad \alpha = 1,2$$

$$(1.3)\qquad \mathbf{m}(\mathbf{u}_s,\mathbf{u},\mathbf{u}_{st},\mathbf{u},s_\alpha)\cdot\frac{\partial \mathbf{g}_\alpha}{\partial \mathbf{v}_\alpha}(\mathbf{v}_\alpha,t) = \mathbf{p}_\alpha(\mathbf{u}_s,\mathbf{u},\mathbf{u}_t,t)\cdot\frac{\partial \mathbf{g}_\alpha}{\partial \mathbf{v}_\alpha}(\mathbf{v}_\alpha,t), \qquad \alpha = 1,2 .$$

In equation (1.2), $\mathbf{v} \equiv \mathbf{v}_1 \oplus \mathbf{v}_2$ is an unknown function of t with values in $\mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}$, where $0 \le N_1, N_2 \le N$. In equation (1.3), $\mathbf{p} = \mathbf{p}_1 \oplus \mathbf{p}_2$ is a given function of the indicated arguments with values in $\mathbb{R}^N \oplus \mathbb{R}^N$. Frequently, $N_\alpha = 0$ $(\alpha = 1,2)$, so that equation (1.2) specifies $\mathbf{u}(s_\alpha,t)$ completely while equation (1.3) is vacuous, or $N_\alpha = N$ so that equation (1.3) specifies the value on $\mathbf{m}$ at (s_α,t) completely while equation (1.2)

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is vacuous. Intermediate values of N_α may arise when, for example, the end of the body is constrained to move along a curve or surface in three-dimensional space. We assume that rank $\frac{\partial \underset{\sim}{q}_\alpha}{\partial \underset{\sim}{v}_\alpha} = N_\alpha$.

Initial conditions for $\underset{\sim}{u}$ are

(1.4) $$\underset{\sim}{u}(s,0) = \underset{\sim}{u}_0(s), \quad \underset{\sim}{u}_t(s,0) = \underset{\sim}{u}_1(s).$$

Equations (1.1) to (1.4) are derived in BROWNE (1976); the corresponding stationary equations, (1.10) to (1.12), below, are derived in ANTMAN (1972, 1976a). The existence and regularity of solutions to the stationary equations is treated in ANTMAN (1976b). A special case of the problem considered here is treated in BROWNE (1977).

We assume the following conditions on the given functions:

(1.5) The functions $\underset{\sim}{A}$, $\underset{\sim}{a}$, $\underset{\sim}{m}$, $\underset{\sim}{n}$, $\underset{\sim}{f}$, and $\underset{\sim}{p}$ are defined for $(\underset{\sim}{u}_s, \underset{\sim}{u}, \underset{\sim}{u}_{st}, \underset{\sim}{u}_t, s, t)$ in an open subset of $[\mathbb{R}^N]^4 \times [s_1, s_2] \times \mathbb{R}^+$, and for each value of these arguments;

(1.6) $\frac{\partial \underset{\sim}{m}}{\partial \underset{\sim}{u}_s}(\underset{\sim}{u}_s, \underset{\sim}{u}, \underset{\sim}{u}_{st}, \underset{\sim}{u}_t, s) \in L(\mathbb{R}^N; \mathbb{R}^N)$ is a positive definite and symmetric transformation on $\mathbb{R}^N$,

(1.7) $\frac{\partial \underset{\sim}{m}}{\partial \underset{\sim}{u}_{st}}(\underset{\sim}{u}_s, \underset{\sim}{u}, \underset{\sim}{u}_{st}, \underset{\sim}{u}_t, s) \in L(\mathbb{R}^N; \mathbb{R}^N)$ is a positive definite (but not necessarily symmetric) transformation on $\mathbb{R}^N$,

(1.8) $\underset{\sim}{A}(\underset{\sim}{u}_t, s) \in L(\mathbb{R}^N; \mathbb{R}^N)$ is a positive definite symmetric transformation on $\mathbb{R}^N$,

(1.9) the functions $\underset{\sim}{A}$ and $\underset{\sim}{a}$ are related by

$$\underset{\sim}{a}(\underset{\sim}{u}, \underset{\sim}{w}, s) = \frac{\partial}{\partial \underset{\sim}{u}} \underset{\sim}{w} \cdot \underset{\sim}{A}(\underset{\sim}{u}, s) \cdot \underset{\sim}{w} \quad .$$

In this paper we construct solutions to equations (1.1) to (1.4) in the neighborhood of a solution $\underset{\sim}{u}^*, \underset{\sim}{v}^*$ of the stationary problem

$$(1.10) \qquad -\frac{\partial}{\partial s}\, \underset{\sim}{m}(\underset{\sim}{u}_s^*, \underset{\sim}{u}^*, \underset{\sim}{0}, \underset{\sim}{0}, s) + \underset{\sim}{m}(\underset{\sim}{u}_s^*, \underset{\sim}{u}^*, \underset{\sim}{0}, \underset{\sim}{0}, s) = \underset{\sim}{f}^*(\underset{\sim}{u}_s^*, \underset{\sim}{u}^*, s)$$

$$(1.11) \qquad \underset{\sim}{u}^*(s_\alpha) = (v_\alpha^*)$$

$$(1.12) \qquad \underset{\sim}{m}(\underset{\sim}{u}_s^*, \underset{\sim}{u}, \underset{\sim}{0}, \underset{\sim}{0}, s_\alpha)\cdot\frac{\partial \underset{\sim}{q}_\alpha^*}{\partial \underset{\sim}{v}_\alpha^*}(\underset{\sim}{v}_\alpha) = \underset{\sim}{p}_\alpha^*(\underset{\sim}{u}_s^*, \underset{\sim}{u}^*)\cdot\frac{\partial \underset{\sim}{q}_\alpha^*}{\partial \underset{\sim}{v}_\alpha^*}(\underset{\sim}{v}_\alpha) .$$

Our main tool is the implicit function theorem. In this section we give an explanation of our notation and the definitions of the Banach spaces employed. In Section 2 we give criteria for the continuity and Fréchet differentiability of functions on these Banach spaces. In Section 3 we study the linearized version of equations (1.1) to (1.4), obtaining an estimate on the decay of solutions as $t \to \infty$. In Section 4 we combine the results of Sections 2 and 3 to obtain solutions to the full nonlinear problem. When these results apply we obtain existence and stability in a single step and obtain the same decay estimate as $t \to \infty$ as obtained for the linear equation. We obtain uniqueness from a similar argument.

We call the stationary problem <u>conservative</u> if there exist real valued functions ψ_0, ψ_1, $\psi_{2\alpha}$ such that

$$(1.13) \qquad \begin{aligned} \underset{\sim}{m}(\underset{\sim}{u}_s, \underset{\sim}{u}, \underset{\sim}{0}, \underset{\sim}{0}, s) &= \frac{\partial}{\partial \underset{\sim}{u}_s}\psi_0(\underset{\sim}{u}_s, \underset{\sim}{u}, s) , \\ \underset{\sim}{n}(\underset{\sim}{u}_s, \underset{\sim}{u}, \underset{\sim}{0}, \underset{\sim}{0}, s) &= \frac{\partial}{\partial \underset{\sim}{u}}\psi_0(\underset{\sim}{u}_s, \underset{\sim}{u}, s) , \end{aligned}$$

$$(1.14) \qquad \begin{aligned} &\frac{\partial^2}{\partial \underset{\sim}{u}_s^2}\psi_1 = \underset{\sim}{0} , \\ &\underset{\sim}{f}^*(\underset{\sim}{u}_s, \underset{\sim}{u}, s) = -\frac{\partial}{\partial \underset{\sim}{u}}\psi_1(\underset{\sim}{u}_s, \underset{\sim}{u}, s) + \frac{\partial^2}{\partial \underset{\sim}{u}\partial \underset{\sim}{u}_s}\psi_1(\underset{\sim}{u}_s, \underset{\sim}{u}, s)\cdot\underset{\sim}{u}_s + \frac{\partial^2}{\partial s\partial \underset{\sim}{u}_s}\psi_1(\underset{\sim}{u}_s, \underset{\sim}{u}, s) , \end{aligned}$$

$$(1.15) \qquad \underset{\sim}{p}(\underset{\sim}{u}_s, \underset{\sim}{u})\cdot\frac{\partial \underset{\sim}{q}_\alpha}{\partial \underset{\sim}{v}_\alpha}(\underset{\sim}{v}_\alpha) = -\frac{\partial}{\partial \underset{\sim}{v}_\alpha}\psi_{2_\alpha}(\underset{\sim}{v}_\alpha) - (-1)^\alpha\frac{\partial}{\partial \underset{\sim}{u}_s}\psi_1(\underset{\sim}{u}_s, \underset{\sim}{u}, s_\alpha)\cdot\frac{\partial \underset{\sim}{q}_\alpha}{\partial \underset{\sim}{v}_\alpha}(\underset{\sim}{v}_\alpha)$$

whenever (1.11) holds. If hypothesis (1.13) holds the material may be called hyperelastic.

Hypotheses (1.14) and (1.15) are adopted to make equation (5.1), below, hold. To illustrate that such relations arise in practice, we consider a rod with ends fixed and subject to a unit (force per deformed length) pressure from above. If we take $\underset{\sim}{u} = (x,y) \in \mathbb{R}^2$ and locate the center of the cross section at s at $x(s)\underset{\sim}{i} + y(s)\underset{\sim}{j}$, we may take $\underset{\sim}{f}(\underset{\sim}{u}_s,\underset{\sim}{u},s) = y_s\underset{\sim}{i} - x_s\underset{\sim}{j}$. The loading is conservative, the potential energy of a deformed configuration being given by the signed area between that configuration and the reference (straight) configuration. Thus we take $\psi_1(\underset{\sim}{u}_s,\underset{\sim}{u},s) = \frac{1}{2}(x_s y - xy_s)$. Then hypothesis (1.14) is valid while with our boundary conditions hypothesis (1.15) reduces to the identity $\underset{\sim}{0} = \underset{\sim}{0}$.

In Section 5 we consider conservative problems. We show that the eigenfunction criterion developed in the previous sections may be replaced by the second variation test for stability. If the second variation of the energy

$$(1.16) \qquad E(\underset{\sim}{u},\underset{\sim}{v}) = \int_{s_1}^{s_2} \psi_0[\underset{\sim}{u}] + \psi_1[\underset{\sim}{u}]ds + \psi_{2\alpha}[\underset{\sim}{v}]\Big|_{\alpha=1}^{2}$$

at $\underset{\sim}{u} = \underset{\sim}{u}^*$ is positive definite then $\underset{\sim}{u}^*$ is stable in the topology of $C^{2+\alpha}$. Although the second variation test has been widely used for over a hundred years, its mathematical validity is often questionable, even in one dimensional problems (KNOPS & WILKES, 1973; KNOPS, 1977). If $\underset{\sim}{u} - \underset{\sim}{u}^*$ is small in the C^1 topology then tne second variation estimates $E(\underset{\sim}{u},\underset{\sim}{v}) - E(\underset{\sim}{u}^*,\underset{\sim}{v}^*)$ in terms of the norm of $\underset{\sim}{u} - \underset{\sim}{u}^*$ in the H_1 topology. A straight forward stability argument is thwarted by these distinct topologies. But we shall show that if the second variation is positive definite then all derivatives of $\underset{\sim}{u}$ occurring in the equation remain close to those of $\underset{\sim}{u}^*$.

1.17. Notation.

We represent elements $\underset{\sim}{a}$, $\underset{\sim}{b}$, of $\mathbb{R}^N$ by lower case, bold face sans-serif Latin letters. We denote the usual inner product of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ on

$\mathbb{R}^N$ by $\underset{\sim}{a}\cdot\underset{\sim}{b}$ and we set $|\underset{\sim}{a}| = \sqrt{\underset{\sim}{a}\cdot\underset{\sim}{a}}$. If $\underset{\sim}{A}$ is a linear transformation of $\mathbb{R}^N$, i.e. $\underset{\sim}{A} \in L(\mathbb{R}^N; \mathbb{R}^N)$, then its value at $\underset{\sim}{a}$ is denoted $\underset{\sim}{A}\cdot\underset{\sim}{a}$ and the value of the quadratic form on $\underset{\sim}{A}$ at $\underset{\sim}{a}$ by $\underset{\sim}{a}\cdot\underset{\sim}{A}\cdot\underset{\sim}{a}$. If $\underset{\sim}{g}$ is a differentiable function on a domain in $\mathbb{R}^n$ then its derivative at $\underset{\sim}{a}$, which is an element of $L(\mathbb{R}^n; \mathbb{R}^N)$, is denoted by $\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{u}}(\underset{\sim}{a})$. Thus $\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{u}}(\underset{\sim}{a})\cdot\underset{\sim}{b}$ denotes the differential of $\underset{\sim}{g}$ at $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$. On the other hand, $\underset{\sim}{c}\cdot\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{u}}(\underset{\sim}{a})$ denotes the element of $L(\mathbb{R}^n; \mathbb{R})$ whose value at $\underset{\sim}{b}$ is $\underset{\sim}{c}\cdot\left(\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{u}}(\underset{\sim}{a})\cdot\underset{\sim}{b}\right)$. We use a similar notation for derivatives of other kinds of functions. We use the same notation for elements of $\mathbb{C}^N$.

If $(s,t) \to \underset{\sim}{u}(s,t)$ is a function defined on $[s_1,s_2] \times \mathbb{R}^+$ we denote $\frac{\partial \underset{\sim}{u}}{\partial s}$ by $\underset{\sim}{u}_s$ and we denote $\frac{\partial \underset{\sim}{u}}{\partial t}$ by $\underset{\sim}{u}_t$. We denote the function $s \to \underset{\sim}{u}(s,t)$ by $\underset{\sim}{u}(\cdot,t)$. If $(\underset{\sim}{u},\underset{\sim}{v},\underset{\sim}{w},s,t) \to \underset{\sim}{g}(\underset{\sim}{u},\underset{\sim}{v},\underset{\sim}{w},s,t)$ is a function defined on $[\mathbb{R}^N]^3 \times [s_1,s_2] \times \mathbb{R}^+$ and the composition $\underset{\sim}{g}(\underset{\sim}{u},\underset{\sim}{u}_s,\underset{\sim}{u}_t,s,t)$ is of interest, we denote this function by $(s,t) \to \underset{\sim}{g}[\underset{\sim}{u}](s,t)$. We use the same notation for other types of compositions.

1.18. Abstract Spaces.

Let $I \subset \mathbb{R}$, $0 < T \leq \infty$ and $Q_T = I \times [0,T]$, or $I \times \mathbb{R}^+$ if $T = \infty$, and let $0 < \alpha < 1$. Then $C^\alpha(I;\mathbb{R}^n)$ is the Banach space of $\mathbb{R}^n$ valued functions on I, with finite norm

$$(1.19) \qquad |\underset{\sim}{u}|_\alpha = \sup_{s \in I} |\underset{\sim}{u}(s)| + \sup_{s_i, t_i \in I} \frac{|\underset{\sim}{u}(s_1) - \underset{\sim}{u}(s_2)|}{|s_1 - s_2|^\alpha},$$

and $C^{\alpha,\frac{1}{2}\alpha}(Q_T;\mathbb{R}^n)$ is the Banach space of $\mathbb{R}^n$ valued functions on Q_T with finite norm

$$(1.20) \qquad |\underset{\sim}{u}|_{\alpha,\frac{\alpha}{2}} = \sup_{(s,t)\in Q_T} |\underset{\sim}{u}(s,t)| + \sup_{(s_i,t_i)\in Q_T} \frac{|\underset{\sim}{u}(s_1,t_1) - \underset{\sim}{u}(s_2,t_2)|}{|s_1 - s_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}}}.$$

We define $C^{2+\alpha}(I;\mathbb{R}^n)$ to be the Banach space of all $\mathbb{R}^n$ valued functions on I, twice differentiable in s, with finite norm

(1.21) $$|\underset{\sim}{u}|_{2+\alpha} = |\underset{\sim}{u}|_\alpha + |\underset{\sim}{u}_s|_\alpha + |\underset{\sim}{u}_{ss}|_\alpha \;;$$

and we define $C^{2+\alpha,1+\frac{1}{2}\alpha}(Q_T;\mathbb{R}^n)$ to be the Banach space of $\mathbb{R}^n$ valued functions on Q_T with finite norm

(1.22) $$|\underset{\sim}{u}|_{2+\alpha,1+\frac{1}{2}\alpha} = |\underset{\sim}{u}|_{\alpha,\frac{1}{2}\alpha} + |\underset{\sim}{u}_s|_{\alpha,\frac{1}{2}\alpha} + |\underset{\sim}{u}_{ss}|_{\alpha,\frac{1}{2}\alpha} + |\underset{\sim}{u}_t|_{\alpha,\frac{1}{2}\alpha} \,.$$

For any real σ we define $C_\sigma^{\alpha,\frac{1}{2}\alpha}$ and $C_\sigma^{2+\alpha,1+\frac{1}{2}\alpha}$ to be the spaces of functions on Q_T with finite norm

(1.23) $$|\underset{\sim}{u}|_{\alpha,\frac{1}{2}\alpha,\sigma} = |\underset{\sim}{v}|_{\alpha,\frac{1}{2}\alpha}\,, \quad |\underset{\sim}{u}|_{2+\alpha,1+\frac{1}{2}\alpha,\sigma} = |\underset{\sim}{v}|_{2+\alpha,1+\frac{1}{2}\alpha}$$

where $\underset{\sim}{v} = e^{\sigma t}\underset{\sim}{u}$. We define the spaces X^α, $X^{\alpha,\frac{1}{2}\alpha}$, etc. to be the closure of the C^∞ functions in the respective spaces C^α, $C^{\alpha,\frac{1}{2}\alpha}$, etc.

Let D denote an open set in either $\mathbb{R}^m \times I$ or $\mathbb{R}^m \times Q_T$ according to the context. We define $A^\alpha(D;\mathbb{R}^n)$ to be the space of $\mathbb{R}^n$ valued functions $\underset{\sim}{g}$ on D such that there exists a constant K and function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $(\underset{\sim}{u}_i, s_i) \in D$

(1.24) $$|\underset{\sim}{g}(\underset{\sim}{u}_1,s_1)| < K \,,$$

(1.25) $$|g(\underset{\sim}{u}_1,s_1) - g(\underset{\sim}{u}_2,s_2)| < K|\underset{\sim}{u}_1 - \underset{\sim}{u}_2| + \omega(|s_1 - s_2|^\alpha) \,,$$

(1.26) $$\lim_{h\to 0^+} \frac{\omega(h)}{h} = 0 \,.$$

We define $A^{\alpha,\frac{1}{2}\alpha}(D,\mathbb{R}^n)$ to be the space of $\mathbb{R}^n$ valued functions $\underset{\sim}{g}$ on D such that there exists a constant K and function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $(\underset{\sim}{u}_i, s_i, t_i) \in D$

$$(1.27)\qquad |g(u_1,s_1,t_1)| < K$$

$$(1.28)\quad |g(u_1,s_1,t_1) - g(u_2,s_2,t_2)| < K|u_1 - u_2| + \omega(|s_1 - s_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}})$$

and equation (1.26) holds. We topologize A^α and $A^{\alpha,\frac{1}{2}\alpha}$ with the norms

$$(1.29)\qquad |g|_{A^\alpha} = \sup_D |g(u,s)| + \sup_D \frac{|g(u_1,s_1) - g(u_2,s_2)|}{|u_1 - u_2| + |s_1 - s_2|^\alpha} ,$$

$$(1.30)\quad |g|_{A^{\alpha,\frac{1}{2}\alpha}} = \sup_D |g(u,s,t)| + \sup_D \frac{g(u_1,s_1,t_1) - g(u_2,s_2,t_2)_{\mathbb{B}}}{|u_1 - u_2| + |s_1 - s_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}}} .$$

We define $A^{1,\alpha}$ to be the subset of A^α such that $\frac{\partial g}{\partial u} \in A^\alpha$, with the obvious norm. We define the spaces $A^{2+\alpha}$, $A^{1,\alpha,\frac{1}{2}\alpha}$, etc. analogously. Finally we define $A_\sigma^{\alpha,\frac{1}{2}\alpha}$, $A_\sigma^{1,\alpha,\frac{1}{2}\alpha}$, etc. to be the subsets of the respective spaces $A^{\alpha,\frac{1}{2}\alpha}$, $A^{1,\alpha,\frac{1}{2}\alpha}$, etc. such that the map $(s,t) \mapsto g(0,s,t)$ is in the space $C_\sigma^{\alpha,\frac{1}{2}\alpha}$ or $C_\sigma^{2+\alpha,1+\frac{1}{2}\alpha}$ as appropriate, with the obvious norm.

2.0. Calculus in $X^{\alpha,\frac{1}{2}\alpha}$.

In this section we give criteria for the continuity and Fréchet diffferentiability of functions defined on $X^{\alpha,\frac{1}{2}\alpha}$ and $X^{\alpha,\frac{1}{2}\alpha}$. Analogous results hold for the other X-spaces.

2.1. Proposition. Let $\underset{\sim}{u} \in C^{\alpha,\frac{1}{2}\alpha}(Q_T;\mathbb{R}^n)$. Then a necessary and sufficient condition that $\underset{\sim}{u} \in X^{\alpha,\frac{1}{2}\alpha}(Q_T;\mathbb{R}^n)$ is that there exist a function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that for $(s_i,t_i) \in Q_T$

$$|\underset{\sim}{u}(s_1,t_1) - \underset{\sim}{u}(s_2,t_2)| < \omega(|s_1 - s_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}}) , \tag{2.2}$$

$$\lim_{h\to 0^+} \frac{\omega(h)}{h} = 0 . \tag{2.3}$$

Proof. First suppose that $\underset{\sim}{u}$ is given and that a function ω satisfying conditions (2.2) and (2.3) exists. Extend $\underset{\sim}{u}$ to all of $\mathbb{R}^2$ with the same modulus of continuity. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^+$ by a C^∞ function with support in $\{(s,t) \in \mathbb{R}^2 : |s| + |t|^{\frac{\alpha}{2}} \le 1\}$, and such that

$$\iint_{\mathbb{R}^2} \phi(s,t)dsdt = 1 .$$

For $\varepsilon > 0$, let

$$\begin{aligned} \underset{\sim}{u}_\varepsilon(s,t) &= \iint_{\mathbb{R}^2} \varepsilon^{-3}\phi(\frac{s-\lambda}{\varepsilon}\ \frac{t-\tau}{\varepsilon^2})\underset{\sim}{u}(\lambda,\tau)d\lambda d\tau \\ &= \iint_{\mathbb{R}^2} \varepsilon^{-3}\phi(\frac{\lambda}{\varepsilon},\frac{\tau}{\varepsilon^2})u(s-\lambda,t-\tau)d\lambda d\tau . \end{aligned}$$

Then $\underset{\sim}{u}_\varepsilon \in C^\infty$ and

$$\begin{aligned} |\underset{\sim}{u} - \underset{\sim}{u}_\varepsilon|_{\alpha,\frac{1}{2}\alpha} &= \sup_{Q_T}|\underset{\sim}{u}(s,t) - \underset{\sim}{u}_\varepsilon(s,t)| \\ &\quad + \sup_{Q_T} \frac{|\underset{\sim}{u}(s_1,t_1) - \underset{\sim}{u}_\varepsilon(s_1,t_1) - \underset{\sim}{u}(s_2,t_2) + \underset{\sim}{u}_\varepsilon(s_2,t_2)|}{|s_1 - s_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}}} \\ &< \omega(\varepsilon) + 2\sup_{h\le\varepsilon}\frac{\omega(h)}{h} . \end{aligned}$$

Thus $\underset{\sim}{u} \in X^{\alpha,\frac{1}{2}\alpha}$.

To prove the converse, suppose that $\underset{\sim}{u} \in C^{\alpha,\frac{1}{2}\alpha}(Q_T;\mathbb{R}^n)$, let

$$(2.4) \qquad \omega(h) = \sup\left\{\frac{|\underset{\sim}{u}(s_1,t_1) - \underset{\sim}{u}(s_2,t_2)|}{|s_1 - s_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}}} : |s_1 - s_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}} \le h\right\},$$

and suppose that

$$\limsup_{h\to 0^+} \frac{\omega(h)}{h} = K > 0 .$$

Let $\underset{\sim}{v} \in C^\infty(Q_T;\mathbb{R}^n)$ and suppose that $\underset{\sim}{v}$ has Lipschitz constant L. Then

$$|\underset{\sim}{u} - \underset{\sim}{v}|_{\alpha,\frac{1}{2}\alpha} \ge \limsup_{h\to 0^+} \sup\left\{\frac{|(\underset{\sim}{u}-\underset{\sim}{v})(s_1,t_1)-(\underset{\sim}{u}-\underset{\sim}{v})(s_2,t_2)|}{|s_1-s_2|^\alpha + |t_1-t_2|^{\frac{\alpha}{2}}} : |s_1-s_2|^\alpha + |t_1-t_2|^{\frac{\alpha}{2}} \le h\right\}$$

$$\ge \limsup_{h\to 0^+} \frac{\omega(h)}{h} - Lh^{(\frac{1}{\alpha}-1)} = K > 0 .$$

Thus $\underset{\sim}{u} \notin X^{\alpha,\frac{1}{2}\alpha}$. $\square$

2.5. Proposition. *Let K be a closed bounded subset of* $X^{\alpha,\frac{1}{2}\alpha}(Q_T;\mathbb{R})$. *Then a necessary condition that K be compact is that there exist a function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $\underset{\sim}{u} \in K$ and $(s_i,t_i) \in Q_T$*

$$(2.5) \qquad |\underset{\sim}{u}(s_1,t_1) - \underset{\sim}{u}(s_2,t_2)| < \omega(|s_1 - s_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}})$$

$$(2.6) \qquad \lim_{h\to 0^+} \frac{\omega(h)}{h} = 0 .$$

If $T < \infty$ then the condition is also sufficient.

Proof. To prove that the condition is necessary, let

$$\omega(h) = \sup_{\underset{\sim}{u}\in K} \omega_{\underset{\sim}{u}}(h)$$

where $\omega_{\underset{\sim}{u}}$ is the function ω defined by equation (2.4), and suppose that

$$\limsup_{h \to 0^+} \frac{\omega(h)}{h} = K > 0 .$$

Choose a sequence $\underset{\sim}{u}_n \in K$ such that $\omega_{\underset{\sim}{u}_n}(\frac{1}{n}) > \frac{K}{2}$. Now for each $(s_i, t_i) \in Q_T$

$$(2.7) \quad |\underset{\sim}{u}_n - \underset{\sim}{u}_m| > Q_{nm}(s_i, t_i) \equiv \frac{|\underset{\sim}{u}_n(s_1,t_1) - \underset{\sim}{u}_m(s_1,t_1) - \underset{\sim}{u}_n(s_2,t_2) + \underset{\sim}{u}_m(s_2,t_2)|}{|s_1 - s_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}}} .$$

By Proposition 2.1, for fixed m there exists a $\delta > 0$ such that

$$\sup_{h<\delta} \frac{1}{h} \omega_{\underset{\sim}{u}_m}(h) < \frac{K}{4} .$$

Then for $\frac{1}{n} < \delta$, $|\underset{\sim}{u}_n - \underset{\sim}{u}_m|_{\alpha, \frac{1}{2}\alpha} > \frac{K}{4}$. Thus the sequence $\underset{\sim}{u}_n$ has no Cauchy subsequence, so K is not compact.

Now suppose $T < \infty$ and that a function ω satisfying conditions (2.5) and (2.6) exists. Let $\underset{\sim}{u}_n$ be a sequence in K. Then by the Ascoli theorem there exists a uniformly Cauchy subsequence, also denoted by $\underset{\sim}{u}_n$. We will show that $\underset{\sim}{u}_n$ is Cauchy in $X^{\alpha, \frac{1}{2}\alpha}$, thereby showing that K is compact. Let $\varepsilon > 0$ and choose δ, $0 < \delta < 2$, so that

$$\sup_{h \leq \delta} \frac{\omega(h)}{h} < \frac{\varepsilon}{4} ,$$

and choose n and m so large that

$$(2.9) \quad \sup_{Q_T} \frac{|\underset{\sim}{u}_n(s,t) - \underset{\sim}{u}_m(s,t)|}{\delta} < \frac{\varepsilon}{4} .$$

Then for $(s_i, t_i) \in Q_T$, if $|s_1 - s_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}} < \delta$ we have

$$Q_{nm}(s_i, t_i) \leq \frac{|\underset{\sim}{u}_n(s_1,t_1) - \underset{\sim}{u}_n(s_2,t_2)|}{|s_1 - s_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}}} + \frac{|\underset{\sim}{u}_m(s_1,t_1) - \underset{\sim}{u}_m(s_2,t_2)|}{|s_1 - s_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}}} < \frac{\varepsilon}{2} .$$

If $|s_1 - s_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}} \geq \delta$ we have

$$Q_{nm}(s_i,t_i) < \frac{|\underset{\sim}{u}_n(s_1,t_1)-\underset{\sim}{u}_m(s_1,t_1)|}{\delta} + \frac{|\underset{\sim}{u}_n(s_2,t_2)-\underset{\sim}{u}_n(s_2,t_2)|}{\delta} < \frac{\varepsilon}{2} .$$

Thus in either case, $Q_{nm}(s_i,t_i) < \frac{\varepsilon}{2}$. Then it follows from (2.8) and the choice of δ that $|\underset{\sim}{u}_n - \underset{\sim}{u}_m|_{\alpha,\frac{1}{2}\alpha} < \varepsilon$. Thus the sequence $\underset{\sim}{u}_n$ is Cauchy. □

Let D be an open set in $\mathbb{R}^m \times Q_T$, and let $E^{\alpha,\frac{1}{2}\alpha}(D;\mathbb{R}^m)$ be the set of all $\underset{\sim}{u} \in X^{\alpha,\frac{1}{2}\alpha}(Q_T;\mathbb{R}^m)$ such that for all $(s,t) \in Q_T(\underset{\sim}{u}(s,t),s,t) \in D$. Clearly $E^{\alpha,\frac{1}{2}\alpha}(D;\mathbb{R}^m)$ is open in $X^{\alpha,\frac{1}{2}\alpha}(Q_T;\mathbb{R}^m)$.

2.9. Proposition. Let $\underset{\sim}{g} \in A^{\alpha,\frac{1}{2}\alpha}(D;\mathbb{R}^n)$. Then for $\underset{\sim}{u} \in E^{\alpha,\frac{1}{2}\alpha}(D;\mathbb{R}^m)$, $\underset{\sim}{g}[\underset{\sim}{u}] \in X^{\alpha,\frac{1}{2}\alpha}(Q_T;\mathbb{R}^n)$. The map $(\underset{\sim}{g},\underset{\sim}{u}) \to \underset{\sim}{g}[\underset{\sim}{u}]$ is continuous on $A^{\alpha,\frac{1}{2}\alpha}(D;\mathbb{R}^n) \times E^{\alpha,\frac{1}{2}\alpha}(D;\mathbb{R}^m)$.

Proof. Let K and $\omega_{\underset{\sim}{g}}$ be the Lipschitz constant and modulus of continuity for $\underset{\sim}{g}$ in equation (1.28). Let $\underset{\sim}{u} \in E^{\alpha,\frac{1}{2}\alpha}(D;\mathbb{R}^m)$ and let ω_u be the modulus of continuity for $\underset{\sim}{u}$ given by Proposition 2.1. Clearly, $\underset{\sim}{g}[\underset{\sim}{u}]$ is continuous. For $(s_i,t_i) \in Q_T$ we have

$$\begin{aligned}
|\underset{\sim}{g}[\underset{\sim}{u}](s_1,t_1)-\underset{\sim}{g}[\underset{\sim}{u}](s_2,t_2)| &< K|\underset{\sim}{u}(s_1,t_1)-\underset{\sim}{u}(s_2,t_2)| + \omega_{\underset{\sim}{g}}(|s_1-s_2|^\alpha + |t_1-t_2|^{\frac{\alpha}{2}}) \\
&< (K\omega_{\underset{\sim}{u}}+\omega_{\underset{\sim}{g}})(|s_1-s_2|^\alpha + |t_1-t_2|^{\frac{\alpha}{2}}) \\
&\equiv \omega_{\underset{\sim}{g}[\underset{\sim}{u}]}(|s_1-s_2|^\alpha + |t_1-t_2|^{\frac{\alpha}{2}}) .
\end{aligned}$$

Clearly, $\lim_{h\to 0^+} \frac{\omega_{\underset{\sim}{g}[\underset{\sim}{u}]}(h)}{h} = 0$, so $\underset{\sim}{g}[\underset{\sim}{u}] \in X^{\alpha,\frac{1}{2}\alpha}$ by Proposition 2.1.

Now let $\underset{\sim}{u}_n \to \underset{\sim}{u}_0$ in $E^{\alpha,\frac{1}{2}\alpha}(D;\mathbb{R}^m)$; we shall show that $\underset{\sim}{g}[\underset{\sim}{u}_n] \to \underset{\sim}{g}[\underset{\sim}{u}_0]$ in $X^{\alpha,\frac{1}{2}\alpha}$. Since the range of a convergent sequence is precompact, by Proposition 2.5 there exists a function ω satisfying conditions (2.5)

and (2.6) for all $\underset{\sim}{u}_n$. Then

$$(2.10)\quad \lim_{n\to\infty} \sup_{Q_T} |(\underset{\sim}{g}[\underset{\sim}{u}_0] - \underset{\sim}{g}[\underset{\sim}{u}_n])(s,t)| < \lim_{n\to\infty} \sup_{Q_T} K|(\underset{\sim}{u}_0 - \underset{\sim}{u}_n)(s,t)| = 0 .$$

Given $\varepsilon > 0$ choose $\delta > 0$ so that

$$\sup_{h<\delta} \frac{\omega(h)}{h} < \frac{\varepsilon}{4} ,$$

and choose n so large that (2.8) holds with $m = 0$. Then, repeating the estimate on $Q_{n0}(s_i,t_i)$ in the proof of Proposition 2.5, for large n we have

$$(2.11)\quad \sup \frac{|(\underset{\sim}{g}[\underset{\sim}{u}_0] - \underset{\sim}{g}[\underset{\sim}{u}_n])(s_1,t_1) - (\underset{\sim}{g}[\underset{\sim}{u}_0] - \underset{\sim}{g}[\underset{\sim}{u}_n])(s_2,t_2)|}{|s_1 - s_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}}} < \frac{\varepsilon}{2} .$$

By (2.10) and (2.11), $\underset{\sim}{g}[\underset{\sim}{u}_n] \to \underset{\sim}{g}[\underset{\sim}{u}_0]$ in $X^{\alpha,\frac{1}{2}\alpha}$. Thus the map $\underset{\sim}{u} \to \underset{\sim}{g}[\underset{\sim}{u}]$ is continuous on $E^{\alpha,\frac{1}{2}\alpha}$.

Now let $\underset{\sim}{f} \in A^{\alpha,\frac{1}{2}\alpha}(D;\mathbb{R}^n)$. We have

$$|\underset{\sim}{f}[\underset{\sim}{u}] - \underset{\sim}{g}[\underset{\sim}{u}]|_{\alpha,\frac{1}{2}\alpha} < \sup_{Q_T} |(\underset{\sim}{f}-\underset{\sim}{g})[\underset{\sim}{u}](s,t)|$$

$$+ \sup_{Q_T} \left\{ \frac{|(\underset{\sim}{f}-\underset{\sim}{g})[\underset{\sim}{u}](s_1,t_1) - (\underset{\sim}{f}-\underset{\sim}{g})[\underset{\sim}{u}](s_2,t_2)|}{|\underset{\sim}{u}(s_1,t_1) - \underset{\sim}{u}(s_2,t_2)| + |s_1-s_2|^\alpha + |t_1-t_2|^{\frac{\alpha}{2}}} \cdot \frac{|\underset{\sim}{u}(s_1,t_1) - \underset{\sim}{u}(s_2,t_2)| + |s_1-s_2|^\alpha + |t_1-t_2|^{\frac{\alpha}{2}}}{|s_1-s_2|^\alpha + |t_1-t_2|^{\frac{\alpha}{2}}} \right\}$$

$$< |\underset{\sim}{f}-\underset{\sim}{g}|_{A^{\alpha,\frac{1}{2}\alpha}} (|\underset{\sim}{u}|_{\alpha,\frac{1}{2}\alpha} + 1) .$$

Thus the maps $\underset{\sim}{g} \to \underset{\sim}{g}[\underset{\sim}{u}]$ are equicontinuous on $A^{\alpha,\frac{1}{2}\alpha}$ for $\underset{\sim}{u}$ in bounded subsets of $\tilde{E}^{\alpha,\frac{1}{2}\alpha}$. Combining our results we have the required continuity of the map $(\underset{\sim}{g},\underset{\sim}{u}) \to \underset{\sim}{g}[\underset{\sim}{u}]$. □

2.21. Proposition. The map $(\underset{\sim}{g},\underset{\sim}{u}) \to \underset{\sim}{g}[\underset{\sim}{u}]$ is continuously Fréchet differentiable in $\underset{\sim}{u}$ on $A^{1,\alpha,\frac{1}{2}\alpha}(D;\mathbb{R}^n) \times E^{\alpha,\frac{1}{2}\alpha}(D;\mathbb{R}^m)$. The Fréchet derivative is given by

$$\text{(2.13)} \qquad \frac{\partial}{\partial \underset{\sim}{u}}(\underset{\sim}{g}[\underset{\sim}{u}])\underset{\sim}{w} = \left(\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{u}}[\underset{\sim}{u}]\right)\cdot \underset{\sim}{w} \quad \text{for} \quad \underset{\sim}{w} \in X^{\alpha,\frac{1}{2}\alpha}(Q_T;\mathbb{R}^m) .$$

Proof. Let $\underset{\sim}{g} \in A^{1,\alpha,\frac{1}{2}\alpha}$ and $\underset{\sim}{u} \in E^{\alpha,\frac{1}{2}\alpha}$. Since $E^{\alpha,\frac{1}{2}\alpha}$ is open, there exists $\varepsilon > 0$ such that if $|\underset{\sim}{w}|_{\alpha,\frac{1}{2}\alpha} < \varepsilon$ then $\underset{\sim}{u} + \underset{\sim}{w} \in E$. For such $\underset{\sim}{w}$ we have

$$\underset{\sim}{g}[\underset{\sim}{u} + \underset{\sim}{w}] - \underset{\sim}{g}[\underset{\sim}{u}] - \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{u}}[\underset{\sim}{u}]\underset{\sim}{w} = \int_0^1 \left(\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{u}}[\underset{\sim}{u} + \lambda \underset{\sim}{w}] - \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{u}}[\underset{\sim}{u}]\right)\underset{\sim}{w}\, d\lambda \equiv o(\underset{\sim}{w}) ,$$

and Proposition 2.9 applied to $\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{u}}$ shows that $\lim\limits_{|\underset{\sim}{w}|_{\alpha,\frac{1}{2}\alpha} \to 0} \frac{o(\underset{\sim}{w})}{|\underset{\sim}{w}|_{\alpha,\frac{1}{2}\alpha}} = 0$. Thus equation (2.13) is correct. By Proposition 2.9, the derivative defined by equation (2.13) is continuous. □

2.14. Proposition. Suppose for each $(s,t) \in Q_T$, $(\underset{\sim}{0},s,t) \in \bar{D}$, and the set $\{\underset{\sim}{u} \in \mathbb{R}^m : (\underset{\sim}{u},s,t) \in D\}$ is starshaped with respect to $\underset{\sim}{u} = \underset{\sim}{0}$. Let $E_\sigma^{\alpha,\frac{1}{2}\alpha}(D;\mathbb{R}^m)$ be the set of all $\underset{\sim}{u} \in X_\sigma^{\alpha,\frac{1}{2}\alpha}(Q_T;\mathbb{R}^m)$ such that $(\underset{\sim}{u}(s,t),s,t) \in D$ for all $(s,t) \in Q_T$. Then for all $\sigma \geq 0$ the map $(\underset{\sim}{g},\underset{\sim}{u}) \to \underset{\sim}{g}[\underset{\sim}{u}]$ is continuously Fréchet differentiable from $A^{1,\alpha,\frac{1}{2}\alpha}(D;\mathbb{R}^n) \times E_\sigma^{\alpha,\frac{1}{2}\alpha}(D;\mathbb{R}^m)$ into $X_\sigma^{\alpha,\frac{1}{2}\alpha}(Q_T;\mathbb{R}^n)$. The Fréchet derivative is given by equation (2.13).

Proof. The space $X_\sigma^{\alpha,\frac{1}{2}\alpha}$ is isomorphic to $X^{\alpha,\frac{1}{2}\alpha}$ upon multiplication by $e^{\sigma t}$; see equation (1.23). Let $\underset{\sim}{g} \in A_\sigma^{\alpha,\frac{1}{2}\alpha}$ and let $\underset{\sim}{f}(\underset{\sim}{u},s,t) = \underset{\sim}{e}^{\sigma t}\underset{\sim}{g}(e^{-\sigma t}\underset{\sim}{u},s,t)$. If $\underset{\sim}{u} \in E_\sigma^{\alpha,\frac{1}{2}\alpha}$ and if $\underset{\sim}{v}(s,t) = e^{\sigma t}\underset{\sim}{u}(s,t)$ then we have $\underset{\sim}{f}[\underset{\sim}{v}](s,t) = e^{\sigma t}\underset{\sim}{g}[\underset{\sim}{u}](s,t)$. Thus we apply Proposition 2.12 of $\underset{\sim}{f}$.

Since D is starshaped with respect to the Q_T-axis and since the map $t \to e^{-\sigma t} \in X^{\frac{1}{2}\alpha}(\mathbb{R}^+; \mathbb{R})$, we have

$$|\underset{\sim}{f}(\underset{\sim}{u}_1, s_1, t_1) - \underset{\sim}{f}(\underset{\sim}{u}_2, s_2, t_2)| = \left|\int_0^1 \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}}(\lambda \underset{\sim}{u}_1, s_1, t_1) - \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}}(\lambda \underset{\sim}{u}_2, s_2, t_2) d\lambda\right|$$

$$\le \int_0^1 \left|\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{u}}(\lambda e^{-\sigma t_1}\underset{\sim}{u}_1, s_1, t_1) - \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{u}}(\lambda e^{-\sigma t_2}\underset{\sim}{u}_2, s_2, t_2)\right| d\lambda$$

$$(2.15) \qquad + |e^{\sigma t_1}\underset{\sim}{g}(\underset{\sim}{0}, s_1, t_1) - e^{\sigma t_2}\underset{\sim}{g}(\underset{\sim}{0}, s_2, t_2)|$$

$$\le K|\underset{\sim}{u}_1 - \underset{\sim}{u}_2| + \omega(|s_1 - s_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}})$$

for some constant K and function ω satisfying (1.26). Thus $\underset{\sim}{f} \in A^{\alpha, \frac{1}{2}\alpha}$. Also

$$\left|\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}}(\underset{\sim}{u}_1, s_1, t_1) - \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}}(\underset{\sim}{u}_2, s_2, t_2)\right|$$

$$(2.16) \qquad = \left|\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{u}}(e^{-\sigma t_1}\underset{\sim}{u}_1, s_1, t_1) - \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{u}}(e^{-\sigma t_2}\underset{\sim}{u}_2, s_2, t_2)\right|$$

$$\le K|\underset{\sim}{u}_1 - \underset{\sim}{u}_2| + \omega(|s_1 - s_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}}) .$$

Thus $\underset{\sim}{f} \in A^{1,\alpha,\frac{1}{2}\alpha}$. Repeating the final calculation in (2.15) and (2.16), we have

$$|\underset{\sim}{f}|_{A^{1,\alpha,\frac{1}{2}\alpha}} < K|\underset{\sim}{g}|_{A_\sigma^{1,\alpha,\frac{1}{2}\alpha}} ,$$

so the map $\underset{\sim}{g} \to \underset{\sim}{f}$ is continuous, and Proposition 2.12 applies. □

Let $E^\alpha(D; \mathbb{R}^m)$, $E^{2+\alpha, 1+\frac{1}{2}\alpha}(D; \mathbb{R}^m)$, etc., be the respective subsets of $X^\alpha(I; \mathbb{R}^m)$, $X^{2+\alpha, 1+\frac{1}{2}\alpha}(Q_T; \mathbb{R}^m)$, etc. such that $(\underset{\sim}{u}(s), s) \in D$ for all $s \in I$ or $(\underset{\sim}{u}(s,t), s, t) \in D$ for all $(s,t) \in Q_T$, whichever condition being appropriate. Propositions 2.9, 2.12, and 2.14 extend to give

2.17. Proposition. The map $(\underset{\sim}{g}, \underset{\sim}{u}) \mapsto \underset{\sim}{g}[\underset{\sim}{u}]$ is continuous on

$A^{\alpha}(D;\mathbb{R}^n) \times E^{\alpha}(D;\mathbb{R}^m)$ (or $A^{2+\alpha,1+\frac{1}{2}\alpha}(D;\mathbb{R}^n) \times E^{2+\alpha,1+\frac{1}{2}\alpha}(D;\mathbb{R}^m)$, etc.) and is continuously Fréchet differentiable on $A^{1,\alpha}(D;\mathbb{R}^n) \times E^{\alpha}(D;\mathbb{R}^m)$ (or $A^{1,2+\alpha,1+\frac{1}{2}\alpha}(D;\mathbb{R}^n) \times E^{2+\alpha,1+\frac{1}{2}\alpha}(D;\mathbb{R}^m)$, etc.). If $\sigma \geq 0$ and if the hypothesis of Proposition 2.14 are valid then the map $(g,u) \mapsto g[u]$ is continuously Fréchet differentiable on $A_{\sigma}^{1,2+\alpha,1+\frac{1}{2}\alpha}(D;\mathbb{R}^n) \times E_{\sigma}^{2+\alpha,1+\frac{1}{2}\alpha}(D;\mathbb{R}^m)$.

2.18. Remark. Natural conditions for continuity and differentiability on the spaces C^{α} are difficult to obtain, as the following example illustrates. Let

$$g(u,s) = \min\{|s|^{\alpha}, |u|\} .$$

Then the map $u \to g[u]$ takes $C^{\alpha}(0,1;\mathbb{R})$ into itself, but is discontinuous. In fact $g[0] = 0$ but, for every constant function $\varepsilon \neq 0$, $|g[\varepsilon]|_{\alpha} > 1$.

3.0. The Linearized Equation.

In this section we consider the linear initial-boundary value problem

$$P\underset{\sim}{w}_{tt} - R\underset{\sim}{w}_t - S\underset{\sim}{w} = \underset{\sim}{f}\,, \tag{3.1}$$

$$\underset{\sim}{w}\big|_{s_\alpha} - \underset{\sim}{q} \in \text{Range } \underset{\sim}{H}\,, \tag{3.2}$$

$$B\underset{\sim}{w} + D\underset{\sim}{w} = \underset{\sim}{p}\,, \tag{3.3}$$

$$\underset{\sim}{w}(\cdot,0) = \underset{\sim}{w}_0,\quad \underset{\sim}{w}_t(\cdot,0) = \underset{\sim}{w}_1\,. \tag{3.4}$$

In equation (3.1), $P \in L(X^\alpha(s_1,s_2;\mathbb{R}^N);X^\alpha(s_1,s_2;\mathbb{R}^N))$ is given by

$$P\underset{\sim}{w}(s) = \underset{\sim}{P}_0(s)\cdot\underset{\sim}{w}(s)\,, \tag{3.5}$$

where $\underset{\sim}{P}_0 \in X^\alpha(s_1,s_2;L(\mathbb{R}^N;\mathbb{R}^N))$ and $\underset{\sim}{P}_0(s)$ is positive definite and symmetric for each s; while R and $S \in L(X^{2+\alpha}(s_1,s_2;\mathbb{R}^N);X^\alpha(s_1,s_2;\mathbb{R}^N))$ are given by

$$R\underset{\sim}{w} = \underset{\sim}{R}_0\cdot\underset{\sim}{w}_{ss} + \underset{\sim}{R}_1\cdot\underset{\sim}{w}_s + \underset{\sim}{R}_2\cdot\underset{\sim}{w} \tag{3.6}$$

$$S\underset{\sim}{w} = \underset{\sim}{S}_0\cdot\underset{\sim}{w}_{ss} + \underset{\sim}{S}_1\cdot\underset{\sim}{w}_s + \underset{\sim}{S}_2\cdot\underset{\sim}{w} \tag{3.7}$$

where $\underset{\sim}{R}_k,\underset{\sim}{S}_k \in X^\alpha(s_1,s_2;L(\mathbb{R}^N;\mathbb{R}^N))$ and $\underset{\sim}{R}_0(s)$ and $\underset{\sim}{S}_0(s)$ are positive definite for each s while $\underset{\sim}{S}_0(s)$ is symmetric. In condition (3.2), $\underset{\sim}{H} = \underset{\sim}{H}_1 \oplus \underset{\sim}{H}_2 \in L(\mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2};\mathbb{R}^N \oplus \mathbb{R}^N)$. We identify $\mathbb{R}^N \oplus \mathbb{R}^N$ with the set of functions $\{s_\alpha\} \to \mathbb{R}^N$, so that condition (3.2) is a restriction on the boundary values of $\underset{\sim}{w}$ analogous to condition (1.2). Rank $H_\alpha = N_\alpha$.

In equation (3.3), $B, D \in L(X^{1+\alpha}(s_1,s_2;\mathbb{R}^N);\mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2})$ are given by

$$B\underset{\sim}{w}(s_\alpha) = \underset{\sim}{B}_0(s_\alpha)\cdot\underset{\sim}{w}_s(s_\alpha) + \underset{\sim}{B}_1(s_\alpha)\cdot\underset{\sim}{w}(s_\alpha)\,, \tag{3.8}$$

$$D\underset{\sim}{w}(s_\alpha) = \underset{\sim}{D}_0(s_\alpha)\cdot\underset{\sim}{w}_s(s_\alpha) + \underset{\sim}{D}_1(s_\alpha)\cdot\underset{\sim}{w}(s_\alpha)\,, \tag{3.9}$$

where $\underset{\sim}{B}_k(s_\alpha)$, $\underset{\sim}{D}_k(s_\alpha) \in L(\mathbb{R}^N;\mathbb{R}^{N_\alpha})$. Rank $\underset{\sim}{B}_0(s_\alpha) = N_\alpha$.

We consider $\underset{\sim}{f}$, $\underset{\sim}{q}$, $\underset{\sim}{p}$, $\underset{\sim}{w}_0$, and $\underset{\sim}{w}_1$ as given functions with

$$
(3.10) \qquad
\begin{aligned}
&\underset{\sim}{f} \in X_\sigma^{\alpha,\frac{1}{2}\alpha}(Q_\infty;\mathbb{C}^N)\ ,\\
&\underset{\sim}{q} \in X_\sigma^{2+\frac{1}{2}\alpha}(\{s_\alpha\}\times\mathbb{R}^+;\mathbb{C}^N\oplus\mathbb{C}^N)\\
&\underset{\sim}{p} \in X_\sigma^{\frac{1}{2}\alpha}(\{s_\alpha\}\times\mathbb{R}^+;\mathbb{C}^{N_1}\oplus\mathbb{C}^{N_2})\\
&\underset{\sim}{w}_0,\ \underset{\sim}{w}_1 \in X^{2+\alpha}(s_1,s_2;\mathbb{C}^N)\ .
\end{aligned}
$$

We seek $\underset{\sim}{w}$ satisfying equations (3.1) to (3.4) with

$$
(3.11) \qquad \underset{\sim}{w} \in Y_0^\alpha(Q_\infty)
$$

where $Y_\sigma^\alpha(Q_T)$ is the space of functions $\underset{\sim}{u} \in X_\sigma^{2+\alpha,1+\frac{1}{2}\alpha}(Q_T;\mathbb{C}^N)$ such that $\underset{\sim}{u}_t \in X^{2+\alpha,1+\frac{1}{2}\alpha}(Q_T;\mathbb{C}^N)$ with norm

$$
|\underset{\sim}{u}|_{Y_\sigma^\alpha} = |\underset{\sim}{u}|_{2+\alpha,1+\frac{1}{2}\alpha,\sigma} + |\underset{\sim}{u}_t|_{2+\alpha,1+\frac{1}{2}\alpha,\sigma}\ .
$$

In addition, we seek an estimate of the form

$$
(3.12) \qquad |\underset{\sim}{w}|_{Y_\sigma^\alpha} < K(|\underset{\sim}{f}|_{\alpha,\frac{1}{2}\alpha,\sigma} + |q|_{2+\frac{1}{2}\alpha,\sigma} + |\underset{\sim}{p}|_{\frac{1}{2}\alpha,\sigma} + |\underset{\sim}{w}_0|_{2+\alpha} + |w_1|_{2+\alpha})\ .
$$

In this section the letter K denotes a generic constant, whose value need not be the same on each occurrence. We permit K to depend on α, σ, $|s_2 - s_1|$, the functions B, D, P, R, and S, and, when we specifically admit the possibility, on the time T. In this section all Banach spaces are complex.

3.13. Compatibility Conditions.

In order that $\underset{\sim}{w}$ satisfying conditions (3.1) to (3.11) exist, the data must satisfy the following compatibility conditions:

$$
(3.14) \qquad
\begin{aligned}
&\underset{\sim}{w}_0|_{s_\alpha} - \underset{\sim}{q}(0) \in \text{Range } \underset{\sim}{H}\ ,\\
&\underset{\sim}{w}_1|_{s_\alpha} - \underset{\sim}{q}_t(0) \in \text{Range } \underset{\sim}{H}\ ,\\
&\underset{\sim}{P}_0^{-1}\cdot(R\underset{\sim}{w}_1 + S\underset{\sim}{w}_0 + \underset{\sim}{f})(s_\alpha,0) \in \text{Range } \underset{\sim}{H}_\alpha,\\
&B\underset{\sim}{w}_1 + D\underset{\sim}{w}_0 = \underset{\sim}{p}(0)\ .
\end{aligned}
$$

We denote by F_σ the set of all $(\underset{\sim}{f}, \underset{\sim}{q}, \underset{\sim}{p}, \underset{\sim}{w}_0, \underset{\sim}{w}_1)$ satisfying conditions (3.10) and (3.14). Then F_σ is a Banach space with norm indicated by the right hand side of inequality (3.12).

3.15. Lemma. Let $(\underset{\sim}{f}, \underset{\sim}{q}, \underset{\sim}{p}, \underset{\sim}{w}_0, \underset{\sim}{w}_1) \in F_\sigma$ be given. Then there exists $w \in Y_0^\alpha$ satisfying conditions (3.2) and (3.4), and there exists a constant K such that estimate (3.12) is valid. If $\underset{\sim}{f}' = \underset{\sim}{f} - P\underset{\sim}{w}_{tt} + R\underset{\sim}{w}_t + s\underset{\sim}{w}$ and $\underset{\sim}{p}' = \underset{\sim}{p} - B\underset{\sim}{w} - D\underset{\sim}{w}$ then $(\underset{\sim}{f}', \underset{\sim}{0}, \underset{\sim}{p}', \underset{\sim}{0}, \underset{\sim}{0}) \in F_\sigma$.

Thus we need only consider data of the form $(\underset{\sim}{f}, \underset{\sim}{0}, \underset{\sim}{p}, \underset{\sim}{0}, \underset{\sim}{0})$.

Proof. Let

$$\underset{\sim}{v}_1(s,t) = [(s_2 - s)\underset{\sim}{q}(s_1,t) + (s - s_1)\underset{\sim}{q}(s_2,t)]/(s_2 - s_1) ,$$

$$\underset{\sim}{v}_2(s,t) = e^{-\sigma t}[\underset{\sim}{w}_0(s) - \underset{\sim}{v}_1(s,0)] ,$$

$$\underset{\sim}{v}_3(s,t) = te^{-\sigma' t}[\underset{\sim}{w}_1(s) - \underset{\sim}{v}_{1t}(s,0) - \underset{\sim}{v}_{2t}(s,0)] \quad \text{where } \sigma' > \sigma.$$

Then $\underset{\sim}{w} = \underset{\sim}{v}_1 + \underset{\sim}{v}_2 + \underset{\sim}{v}_3$ is the desired function. It is clear that an estimate of the form (3.12) holds and that $\underset{\sim}{f}'$ and $\underset{\sim}{p}'$ satisfy (3.10). We observe that since $\underset{\sim}{f}$ and $\underset{\sim}{p}$ satisfy (3.14) we have $\underset{\sim}{f}'(s_\alpha,0) = \underset{\sim}{0}$ and $\underset{\sim}{p}'(s_\alpha,0) = \underset{\sim}{0}$, so that (3.14) is satisfied by $(\underset{\sim}{f}', \underset{\sim}{0}, \underset{\sim}{p}', \underset{\sim}{0}, \underset{\sim}{0})$. □

3.16. Construction of $\underset{\sim}{w}$.

3.17. Theorem. Let $(\underset{\sim}{f},\underset{\sim}{q},\underset{\sim}{p},\underset{\sim}{w}_0,\underset{\sim}{w}_1) \in F_\sigma$ be given. Then for each $T < \infty$ there exists a $\underset{\sim}{w} \in Y_\sigma^\alpha(Q_T)$ satisfying equations (3.1) to (3.4). There exists a constant K, depending on T, such that

$$|\underset{\sim}{w}|_{Y_\sigma^\alpha(Q_T)} < K|(\underset{\sim}{f},\underset{\sim}{q},\underset{\sim}{p},\underset{\sim}{w}_0,\underset{\sim}{w}_1)|_{F_\sigma} . \tag{3.18}$$

Proof. The spaces $Y_\sigma^\alpha(Q_T)$ are clearly equivalent for $T < \infty$, thus we consider only $\sigma = 0$. By Lemma 3.15, it is sufficient to consider data of the form $(\underset{\sim}{f},\underset{\sim}{0},\underset{\sim}{p},\underset{\sim}{0},\underset{\sim}{0})$. Let $\underset{\sim}{v} = W_t(\underset{\sim}{f},\underset{\sim}{p})$ be the solution of the parabolic system

$$
(3.19)\qquad
\begin{aligned}
P\underset{\sim}{v}_t - R\underset{\sim}{v} &= \underset{\sim}{f} \,, \\
\underset{\sim}{v}\big|_{s_\alpha} &\in \text{Range } \underset{\sim}{H} \,, \\
B\underset{\sim}{v} &= \underset{\sim}{p} \,, \\
\underset{\sim}{v}(s,0) &= \underset{\sim}{0} \,;
\end{aligned}
$$

and let

$$
(3.20)\qquad W(\underset{\sim}{f},\underset{\sim}{p})(s,t) = \int_0^t W_t(\underset{\sim}{f},\underset{\sim}{p})(s,\tau)\,d\tau \ .
$$

Equation (3.19) is parabolic because $\underset{\sim}{P}_0$ and $\underset{\sim}{R}_0$ are positive definite while $\underset{\sim}{P}_0$ is symmetric. From the theory of parabolic systems (LADYŽENSKAYA & SOLONNIKOV, 1967), equation (3.19) has a unique solution in $C^{2+\alpha,1+\frac{1}{2}\alpha}(Q_T;\mathbb{C}^N)$ and there exists a constant K, such that

$$
(3.21)\qquad |\underset{\sim}{v}|_{2+\alpha,1+\frac{1}{2}\alpha} < K(|\underset{\sim}{f}|_{\alpha,\frac{1}{2}\alpha} + |\underset{\sim}{p}|_{\frac{\alpha}{2}}) \ .
$$

The constant K depends on α, $s_2 - s_1$, T, $\underset{\sim}{H}$, $\underset{\sim}{B}$, the Holder norms of the functions $\underset{\sim}{P}_0$ and $\underset{\sim}{R}_k$, and on the minimum eigenvalues of $\underset{\sim}{P}_0$ and of the symmetric part of $\underset{\sim}{R}_0$. If we approximate these functions by C^∞ functions, and likewise approximate $\underset{\sim}{f}$ and $\underset{\sim}{p}$, the solution to equation (3.19) is C^∞ and, by (3.21), approximates $\underset{\sim}{v}$ in $C^{2+\alpha,1+\frac{1}{2}\alpha}$. Thus we see that $\underset{\sim}{v} \in X^{2+\alpha,1+\frac{1}{2}\alpha}(Q_T;\mathbb{C}^N)$. From (3.20) and (3.21), we have

$$
(3.22)\qquad
\begin{aligned}
&\|W_t\|_{L(X^{\alpha,\frac{1}{2}\alpha}(Q_T;\mathbb{C}^N)\times X^{\frac{1}{2}\alpha}(\{s_\alpha\}\times\mathbb{R}^+;\,\mathbb{C}^{N_1}\oplus\mathbb{C}^{N_2});\,X^{2+\alpha,1+\frac{1}{2}\alpha}(Q_T;\mathbb{C}^N))} < K \,, \\
&\|W\|_{L(X^{\alpha,\frac{1}{2}\alpha}(Q_T;\mathbb{C}^N)\times X^{\frac{1}{2}\alpha}(\{s_\alpha\}\times\mathbb{R}^+;\,\mathbb{C}^{N_1}\oplus\mathbb{C}^{N_2});\,Y_0^\alpha(Q_T))} < K \ .
\end{aligned}
$$

We observe that the solution of equations (3.1) to (3.4) may be characterized as a fixed point of the transformation on $Y_0^\alpha(Q_T)$:

(3.23) $$\underset{\sim}{w} \to W(\underset{\sim}{f} + S\underset{\sim}{w}, \underset{\sim}{p} - D\underset{\sim}{w}) = W(\underset{\sim}{f}, \underset{\sim}{p}) + WJ\underset{\sim}{w} ,$$

where for $w \in Y^{\alpha}(Q_T)$

$$J\underset{\sim}{w} = (S\underset{\sim}{w}, -D\underset{\sim}{w}) \in X^{\alpha, \frac{1}{2}\alpha}(Q_T; \mathbb{C}^N) \times X^{\frac{1}{2}\alpha}(\{s_\alpha\} \times \mathbb{R}^+; \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2}) .$$

To construct such a fixed point, let $\underset{\sim}{w}^0 \in Y^{\alpha}(Q_T)$ and define

$$\underset{\sim}{w}^n = W(\underset{\sim}{f}, \underset{\sim}{p}) + WJ\underset{\sim}{w}^{n-1}, \qquad n = 1,2,\ldots .$$

A simple calculation gives

(3.24) $$\underset{\sim}{w}^n = \sum_{k=0}^{n-1} (WJ)^k W(\underset{\sim}{f}, \underset{\sim}{p}) + (WJ)^n \underset{\sim}{w}^0 .$$

Let us show inductively that

(3.25) $$\|(WJ)^n\| \le \|W_t\|^n \|J\|^n T^n / n!, \qquad n = 0,1,2,\ldots ,$$

(3.26) $$\|W_t J (WJ)^n\| \le \|W_t\|^n \|J\|^n T^{n-1}/(n-1)!, \qquad n = 1,2,\ldots .$$

Inequality (3.25) is true when $n = 0$. Since $\|W_t J\| < \|W_t\| \|J\|$, inequality (3.26) for $n = k + 1$ follows from inequality (3.25) for $n = k$. Integrating inequality (3.26) for $n = k + 1$ gives inequality (3.25) for $n = k + 1$. Thus inequalities (3.25) and (3.26) are proven.

Then we may pass to the limit in equation (3.24) as $n \to \infty$ to show that $\underset{\sim}{w}^n$ converges to

(3.27) $$\underset{\sim}{w} = \sum_{k=0}^{\infty} (WJ)^k W(\underset{\sim}{f}, \underset{\sim}{p}) .$$

Inequality (3.25) shows that the sum converges absolutely in $Y^{\alpha}(Q_T)$ and gives the estimate (3.18). Since $\underset{\sim}{w}$ is the limit of $\underset{\sim}{w}^n$, and since the transformation (3.23) is continuous on $Y^{\alpha}(Q_T)$, $\underset{\sim}{w}$ is a fixed point of the transformation (3.23). Since the limit is independent of the choice of $\underset{\sim}{w}^0$, the solution is unique. □

From the proof of Theorem 3.17 we have

3.28. Corollary. *Let* $(\underset{\sim}{f}, \underset{\sim}{0}, \underset{\sim}{p}, \underset{\sim}{0}, \underset{\sim}{0}) \in F_\sigma$ *be given and let* W *be defined by* (3.19) *and* (3.20). *Then the solution* $\underset{\sim}{w}$ *of equations* (3.1) *to* (3.4) *is given by equation* (3.27).

3.29. Asymptotic Behavior of $\underset{\sim}{w}$.

In this section we obtain an estimate on $|\underset{\sim}{w}(\cdot,t)|_{2+\alpha}$ as $t \to \infty$. We begin by examining $\underset{\sim}{v} \equiv (WJ)^n W(\underset{\sim}{f},\underset{\sim}{p})$. Since $\underset{\sim}{v}$ is the solution of a linear evolution equation, $\underset{\sim}{v}$ can grow at most exponentially in t. By (3.25), $\underset{\sim}{v}$ is Hölder continuous. Thus we may construct $\underset{\sim}{v}$ by Laplace transforms. Doing so, we obtain

$$\text{(3.30)} \quad \underset{\sim}{v}(\cdot,t) = \frac{1}{2\pi i}\int_0^t \int_{a-i\infty}^{a+i\infty} e^{(t-\tau)\zeta}(\tilde{W}(\zeta)J)^n\tilde{W}(\zeta)(\underset{\sim}{f}(\cdot,\tau),\underset{\sim}{p}(\tau))\,d\zeta\, d\tau\ ,$$

where $\underset{\sim}{w} = \tilde{W}(\zeta)(\underset{\sim}{f},\underset{\sim}{p})$ is the solution of

$$\text{(3.31)} \quad \begin{aligned} &(\zeta^2 P - \zeta R)\underset{\sim}{w} = \underset{\sim}{f}\ , \\ &\underset{\sim}{w}\big|_{S_\alpha} \in \text{Range } \underset{\sim}{H}\ , \\ &B\underset{\sim}{w} = \underset{\sim}{p}\ , \end{aligned}$$

and we now view J as an element of $L(X^{2+\alpha}(s_1,s_2;\mathbb{C}^N);$ $X^{\alpha}(s_1,s_2;\mathbb{C}^N) \times \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2})$. Then $\tilde{W}$ is an operator valued analytic function of ζ.

We observe that $\tilde{W}(\zeta)(\underset{\sim}{f},\underset{\sim}{0}) = P(\zeta I - P^{-1}R)^{-1}$, where the inverse is taken among the functions satisfying homogeneous boundary data in equation (3.31). If $\underset{\sim}{v}$ is the solution of equation (3.19) with homogeneous $\underset{\sim}{f}$ and $\underset{\sim}{p}$ but with $\underset{\sim}{v}(s,0) = \underset{\sim}{v}_0(s)$ then the Schauder estimates for parabolic systems (LADYZENSKAYA & SOLONNIKOV, 1967) give

$$|\underset{\sim}{v}(\cdot,t)|_{j+\alpha} < K|\underset{\sim}{v}_0|_\alpha\, t^{-\frac{1}{2}j}\ , \qquad j = 0,2\ .$$

These inequalities and that C^∞ is dense in X^α are known to imply (FRIEDMAN, 1969; PAZY, 1974) that for some positive constants a, and b and for

$$\zeta \in J \equiv \{\zeta \in \mathbb{C} : \text{Re}(\zeta) \geq a - b|\text{Im}(\zeta)|\}$$

$$|(\zeta I - P^{-1}R)^{-1}f|_{j+\alpha} < K|\zeta|^{\frac{j-2}{2}}|\underset{\sim}{f}|_\alpha, \qquad j = 0,2\ .$$

Thus

$$(3.32)\qquad |\tilde{W}(\zeta)(\underset{\sim}{f},\underset{\sim}{0})|_{j+\alpha} < K|\zeta|^{\frac{j-4}{2}} |\underset{\sim}{f}|_{\alpha}, \quad \zeta \in J, \quad j = 0,2 .$$

We shall obtain slightly weaker estimates for non-zero $\underset{\sim}{p}$.

Choose $\underset{\sim}{T} \in L(\mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2};\mathbb{C}^N \oplus \mathbb{C}^N)$ such that $\underset{\sim}{B}_0 \cdot \underset{\sim}{T} = \underset{\sim}{I}$, which we may do because $\underset{\sim}{B}_0$ is of full rank. Let ϕ be a smooth function on $\mathbb{R} \times \mathbb{R}^+$ satisfying

$$\phi(x,y) = 1 \quad \text{for} \quad x < \frac{1}{2}y^{-\frac{1}{2}} ,$$

$$\phi(x,y) = 0 \quad \text{for} \quad x > y^{-\frac{1}{2}} ,$$

$$|\phi(\cdot,y)|_{j+\alpha} < Ky^{-j/2} , \quad j = 0,1,2 .$$

Let

$$(\tilde{W}_1(\zeta)\underset{\sim}{p})(s) = \sum_{\alpha=1}^{2} \phi(s - s_\alpha,|\zeta|)[s - s_\alpha + (-1)^\alpha|\zeta|^{\frac{1}{2}}(s - s_\alpha)^2]\underset{\sim}{T}_\alpha \cdot \underset{\sim}{p}_\alpha/\zeta .$$

Observe that for sufficiently large $|\zeta|$, $\zeta BW_1(\zeta)\underset{\sim}{p} = \underset{\sim}{p}$ and that

$$(3.33)\qquad |\tilde{W}_1(\zeta)\underset{\sim}{p}|_{j+\alpha} < K|\zeta|^{\frac{j-3}{2}} |\underset{\sim}{p}|, \quad j = 0,2, |\zeta| > 4/(s_2 - s_1)^2 .$$

Then

$$(3.34)\qquad \tilde{W}(\zeta)(\underset{\sim}{f},\underset{\sim}{p}) = \tilde{W}(\zeta)(\underset{\sim}{f} - (\zeta^2P - \zeta R)W_1(\zeta)\underset{\sim}{p}) + W_1(\zeta)\underset{\sim}{p} .$$

Taking a larger if necessary, we have from (3.32), (3.33), and (3.34)

$$(3.35)\qquad |\tilde{W}(\zeta)(\underset{\sim}{f},\underset{\sim}{p})|_{j+\alpha} < K|\zeta|^{\frac{j-3}{2}} (|\underset{\sim}{f}|_\alpha + |\underset{\sim}{p}|) , \quad \zeta \in J, \quad j = 0,2 .$$

It follows that

$$(3.36)\qquad |(\tilde{W}(\zeta)J)^n W(\zeta)(\underset{\sim}{f},\underset{\sim}{p})|_{j+\alpha} < K^{n+1}|\zeta|^{\frac{j-n-3}{2}} (|\underset{\sim}{f}|_\alpha + |\underset{\sim}{p}|) .$$

Then we may shift the path of integration with respect to ζ in (3.30) to the path ∂J. Let us further shift the path of integration in (3.30)

to a contour I agreeing with ∂J for large $|\zeta|$ but such that $|\zeta|^{\frac{1}{2}} > K$ on I, where K is the constant in inequality (3.36). Then if $\underset{\sim}{w}$ is given by equation (3.27) we have

$$(3.37) \quad \underset{\sim}{w}(\cdot,t) = \frac{1}{2\pi i}\int_0^t \int_I \sum_{k=0}^{\infty} e^{(t-\tau)\zeta}(W(\zeta)J)^k W(\zeta)(\underset{\sim}{f}(\cdot,\tau),\underset{\sim}{p}(\tau))d\zeta d\tau ,$$

the sum converging absolutely on I.

Let $\tilde{\underset{\sim}{w}} = \tilde{Z}(\zeta)(\underset{\sim}{f},\underset{\sim}{p})$ be the solution of

$$(3.38) \quad \begin{aligned} &(\zeta^2 P - \zeta R - S)\tilde{\underset{\sim}{w}} = \underset{\sim}{f} , \\ &\tilde{\underset{\sim}{w}}|_{S_\alpha} \in \text{Range } \underset{\sim}{H} , \\ &(\zeta B + D)\tilde{\underset{\sim}{w}} = \underset{\sim}{p} . \end{aligned}$$

Then $\tilde{Z}$ is an operator valued analytic function of ζ. One can show, in a fashion analogous to showing that $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$ for $\|A\| < 1$, that for $\zeta \in J$ and $|\zeta|^{\frac{1}{2}} > K$

$$(3.39) \quad \sum_{k=0}^{\infty} (\tilde{W}(\zeta)J)^k \tilde{W}(\zeta) = \tilde{Z}(\zeta) .$$

Combining our results, we have

3.40. Lemma. Let $(\underset{\sim}{f},\underset{\sim}{0},\underset{\sim}{p},\underset{\sim}{0},\underset{\sim}{0}) \in F_\sigma$ be given. Then the solution of equations (3.1) to (3.4) is given by

$$(3.41) \quad \underset{\sim}{w}(\cdot,t) = \frac{1}{2\pi i}\int_0^t \int_I e^{(t-\tau)\zeta}\tilde{Z}(\zeta)(\underset{\sim}{f}(\cdot,\tau),\underset{\sim}{p}(\tau))d\zeta d\tau ,$$

where $\tilde{Z}$ is defined by equations (3.38). There exists a constant K such that for $\zeta \in J$ and $|\zeta| > K$

$$(3.42) \quad |\tilde{Z}(\zeta)(\underset{\sim}{f},\underset{\sim}{p})|_{j+\alpha} < K|\zeta|^{\frac{j-3}{2}}(|\underset{\sim}{f}|_\alpha + |\underset{\sim}{p}|), \qquad j = 0,2 .$$

To obtain (3.41) we estimate (3.39) by (3.36) and sum the geometric series.

3.43. Lemma. Suppose $\tilde{Z}$ is analytic for $\text{Re}(\zeta) > -\sigma_0$. Let $\sigma < \sigma_0$ and $(\underset{\sim}{f},\underset{\sim}{g},\underset{\sim}{p},\underset{\sim}{w}_0,\underset{\sim}{w}_1) \in F_\sigma$ be given and let $\underset{\sim}{w}$ be the solution of equations (3.1)

to (3.4). Then there exists a constant K such that for all $t > 0$

$$(3.44)\qquad \begin{aligned} |e^{\sigma t}\underset{\sim}{w}(\cdot,t)|_{2+\alpha} &< K|(\underset{\sim}{f},\underset{\sim}{g},\underset{\sim}{p},\underset{\sim}{w}_0,\underset{\sim}{w}_1)|_{F_\sigma}, \\ |e^{\sigma t}\underset{\sim}{w}_t(\cdot,t)|_{2+\alpha} &< K|(\underset{\sim}{f},\underset{\sim}{g},\underset{\sim}{p},\underset{\sim}{w}_0,\underset{\sim}{w}_1)|_{F_\sigma}. \end{aligned}$$

Proof. By Lemma 3.15, it is sufficient to consider data of the form $(\underset{\sim}{f},\underset{\sim}{0},\underset{\sim}{p},\underset{\sim}{0},\underset{\sim}{0}) \in F_\sigma$. Let I be the contour in (3.41), and deform I into a contour $I' = I_1' \cup I_2'$ where $|\zeta| < K$ on I_1' and $-\sigma_0 < -\sigma_1 \equiv \sup_{I_1'} \mathrm{Re}(\zeta) < -\sigma$ while $|\zeta| \geq K$ and $\mathrm{Re}(\zeta) = a - b\,\mathrm{Im}(\zeta)$ on I_2'. Let $\phi_1 : \mathbb{R}^+ \to [0,1]$ be a smooth decreasing function such that $\phi_1(t) = 1$ on $[0,2]$ and $\phi_1(t) = 0$ on $[3,\infty)$. Let $\phi_2 = 1 - \phi_1$. For $t > 0$ let T be the first integer greater than t, then by (3.41)

$$\begin{aligned} \underset{\sim}{w}(\cdot,t) &= \frac{1}{2\pi i}\int_0^t \int_{I'} e^{(t-\tau)\zeta}\tilde{Z}(\zeta)[\phi_1(T-\tau) + \phi_2(T-\tau)]\cdot(\underset{\sim}{f}(\cdot,\tau),\underset{\sim}{p}(\tau))\,d\zeta\, d\tau \\ &= \underset{\sim}{w}_1(\cdot,t,T) + \underset{\sim}{w}_2(\cdot,t,T). \end{aligned}$$

The function $\underset{\sim}{w}_1$ is the solution of equations (3.1) to (3.4) with data estimated by $\underset{\sim}{f}$ and $\underset{\sim}{p}$ and supported in $[T-3,T]$; therefore inequalities (3.44) are valid for $\underset{\sim}{w}_1$ by Theorem 3.17.

By (3.41)

$$\begin{aligned} \underset{\sim}{w}_2(\cdot,t,T) &= \frac{1}{2\pi i}\int_0^{t-1}\left(\int_{I_1'} + \int_{I_2'}\right) e^{(t-\tau)\zeta}\tilde{Z}(\zeta)\phi_2(T-\tau)\cdot(\underset{\sim}{f}(\cdot,\tau),\underset{\sim}{p}(\tau))\,d\zeta\, d\tau \\ &\equiv \underset{\sim}{w}_{21}(\cdot,t,T) + \underset{\sim}{w}_{22}(\cdot,t,T). \end{aligned}$$

We have

$$(3.45)\qquad \begin{aligned} |e^{\sigma t}\underset{\sim}{w}_{21}(\cdot,t,T)|_{2+\alpha} &\leq \frac{1}{2\pi}\int_0^{t-1}\int_{I_1'} |e^{(t-\tau)(\zeta+\sigma)}\tilde{Z}(\zeta)e^{\sigma\tau}(\underset{\sim}{f}(\cdot,\tau),\underset{\sim}{p}(\tau))|_{2+\alpha}\,d\zeta\, d\tau \\ &< K|(\underset{\sim}{f},\underset{\sim}{0},\underset{\sim}{p},\underset{\sim}{0},\underset{\sim}{0})|_{F_\sigma}. \end{aligned}$$

Using (3.42), for $t > 1$ we have

$$|e^{\sigma t}\underset{\sim}{w}_{22}(\cdot,t,T)|_{2+\alpha} < \frac{1}{2\pi}\int_0^{t-1}\int_{I_2'} |e^{(t-\tau)(\zeta+\sigma)}\tilde{Z}(\zeta)e^{\sigma\tau}(\underset{\sim}{f}(\cdot,\tau),\underset{\sim}{p}(\tau))|_{2+\alpha}\,d\zeta\, d\tau$$

$$< K|(\underset{\sim}{f},\underset{\sim}{0},\underset{\sim}{p},\underset{\sim}{0},\underset{\sim}{0})|_{F_\sigma}\int_0^{t-1}(t-\tau)^{-\frac{1}{2}}e^{(t-\tau)(\sigma-\sigma_1)}\,d\tau \tag{3.46}$$

$$K|(\underset{\sim}{f},\underset{\sim}{0},\underset{\sim}{p},\underset{\sim}{0},\underset{\sim}{0})|_{F_\sigma}\,.$$

Combining (3.45), (3.46), and our estimate on $\underset{\sim}{w}_1$, we have the desired estimate on $\underset{\sim}{w}(\cdot,t)$ in (3.44). The estimate on $\underset{\sim}{w}_t(\cdot,t)$ is obtained in the same way. □

3.47. Theorem. Let σ_0 be as in Lemma 3.43, and let $\sigma < \sigma_0$. Let $(\underset{\sim}{f},\underset{\sim}{q},\underset{\sim}{p},\underset{\sim}{w}_0,\underset{\sim}{w}_1) \in F_\sigma$ be given and let $\underset{\sim}{w}$ be the solution to equations (3.1) to (3.4). Then there exists a constant K such that

$$|\underset{\sim}{w}|_{Y_\sigma(Q_\infty)} < K|(\underset{\sim}{f},\underset{\sim}{p},\underset{\sim}{q},\underset{\sim}{w}_0,\underset{\sim}{w}_1)|_{F_\sigma}\,.$$

Proof. One obtains an equivalent norm on the Hölder spaces if one restricts oneself to $|t_1 - t_2| \leq 1$ in the Hölder quotients. Thus it is sufficient to estimate $\underset{\sim}{w}$ on each interval $n \leq t \leq n+2$. If we let $\underset{\sim}{w}_n'(s,t) = \underset{\sim}{w}(s,t+n)$ and make similar definitions for $\underset{\sim}{f}_n'$, $\underset{\sim}{q}_n'$, and $\underset{\sim}{p}_n'$, then, by Theorem 3.17,

$$|\underset{\sim}{w}_n'|_{Y_\sigma^\alpha(Q_2)} < K|(\underset{\sim}{f}_n',\underset{\sim}{q}_n',\underset{\sim}{p}_n',\underset{\sim}{w}(\cdot,n),\underset{\sim}{w}_t(\cdot,n))|_{F_0}\,. \tag{3.48}$$

Multiplying (3.48) by $e^{\sigma n}$ and using Lemma 3.43, we obtain the desired estimate. □

3.49. Singularities of $\tilde{Z}$.

Contrary to the experience with first order evolution equations, the "resolvent" $\tilde{Z}$ is not generally meromorphic in $\mathbb{C}$. But we do have:

3.50. Theorem. Let

$$-\sigma_0 = \inf\{\sigma \in \mathbb{R} : \det[\zeta\underset{\sim}{R}_0(s) + \underset{\sim}{S}_0(s)] \neq 0 \text{ for } s_1 \leq s \leq s_2 \text{ and } \mathrm{Re}(\zeta) > \sigma\}\,. \tag{3.51}$$

Then $\sigma_0 > 0$, and $\tilde{Z}$ is meromorphic in ζ for $\mathrm{Re}(\zeta) > -\sigma_0$. If

$\mathrm{Re}(\zeta) > -\sigma_0$ <u>and</u> $\tilde{Z}$ <u>has a pole at</u> ζ_0 <u>then there exists a finite dimensional, non-trivial vector space of functions</u> $\underset{\sim}{w}$ <u>such that</u>

$$
(3.52) \qquad \begin{aligned}
&(\zeta_0^2 P - \zeta_0 R - S)\underset{\sim}{w} = \underset{\sim}{0} , \\
&w|_{s_\alpha} \in \text{Range } \underset{\sim}{H} , \\
&(\zeta_0 B + D)\underset{\sim}{w} = \underset{\sim}{0} .
\end{aligned}
$$

Thus we may verify the hypothesis of Theorem 3.47 by solving the eigenvalue problem (3.52).

Proof. Since the matrices $\underset{\sim}{R}_0(s)$ and $\underset{\sim}{S}_0(s)$ are positive definite while $\underset{\sim}{S}_0(s)$ is symmetric, the determinant in (3.51) is non-zero for $\mathrm{Re}(\zeta) \geq 0$. Since these matrices are continuous in s, we have $\sigma_0 > 0$.

The solution to (3.38) is given by

$$
(3.53) \qquad \underset{\sim}{w}(s) = \tilde{Z}(\zeta)(\underset{\sim}{f},\underset{\sim}{p})(s) = \int_{s_1}^{s_2} \underset{\sim}{G}_1(s,\xi,\zeta)\cdot \underset{\sim}{f}(\xi)d\xi + \underset{\sim}{G}_2(s,\zeta)\cdot \underset{\sim}{p}
$$

for some Green's functions $\underset{\sim}{G}_1$ and $\underset{\sim}{G}_2$ which are meromorphic in ζ whenever the determinant in (3.51) does not vanish, thus for $\mathrm{Re}(\zeta) > -\sigma_0$. We apply Morera's theorem to show that $\tilde{Z}$ is meromorphic: If $(\zeta - \zeta_0)^n \underset{\sim}{G}_k$ is analytic in ζ in a simply connected neighborhood of ζ_0 then we integrate $(\zeta - \zeta_0)^n \tilde{Z}(\zeta)(\underset{\sim}{f},\underset{\sim}{p})$ about a closed contour and use (3.53) to show that the integral vanishes.

Now suppose that $\tilde{Z}$ has a pole at ζ_0, and that

$$
\|\tilde{Z}(\zeta)\|_{L(X^\alpha \times \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2};X^{2+\alpha})} = O(|\zeta - \zeta_0|^{-n}) \qquad (\zeta \to \zeta_0) .
$$

Then there exist $\underset{\sim}{f}$ and $\underset{\sim}{p}$ such that for small $\varepsilon > 0$

$$
\underset{\sim}{w} = \frac{1}{2\pi i} \int_{|\zeta-\zeta_0|=\varepsilon} (\zeta - \zeta_0)^{n-1} \tilde{Z}(\zeta)(\underset{\sim}{f},\underset{\sim}{p})d\zeta \neq \underset{\sim}{0} .
$$

Then, since $\zeta^2 P - \zeta R - S \in L(X^{2+\alpha};X^\alpha)$ is a smooth function of ζ,

$$(\zeta_0^2 P - \zeta_0 R - S)\underset{\sim}{w} = \frac{1}{2\pi i} \int_{|\zeta-\zeta_0|=\varepsilon} (\zeta_0^2 P - \zeta_0 R - S)(\zeta - \zeta_0)^{n-1} \tilde{Z}(\zeta)(\underset{\sim}{f},\underset{\sim}{p})\,d\zeta$$

$$= \frac{1}{2\pi i} \int_{|\zeta-\zeta_0|=\varepsilon} [(\zeta^2 P - \zeta R - S)(\zeta - \zeta_0)^{n-1} + 0(|\zeta - \zeta_0|^n)] \cdot \tilde{Z}(\zeta)(\underset{\sim}{f},\underset{\sim}{p})\,d\zeta$$

$$= 0(\varepsilon) \qquad (\varepsilon \to 0)\ .$$

Since ε may be arbitrarily small, $(\zeta_0^2 P - \zeta_0 R - S)\underset{\sim}{w} = \underset{\sim}{0}$. Similarly, $(\zeta_0 B + D)\underset{\sim}{w} = \underset{\sim}{0}$. Thus equations (3.52) have non-trivial solutions. But solutions to (3.52) coincide with solutions to the eigenvalue problem $(\lambda P - \zeta_0 R - S)\underset{\sim}{w} = \underset{\sim}{0}$ with ζ_0 fixed at $\lambda = \zeta_0^2$; therefore the solutions to (3.52) are finite dimensional. □

4.0. The Nonlinear Equation.

Let $\underset{\sim}{u}^*$, $\underset{\sim}{v}^*$ be a solution to the equilibrium equations (1.14) to (1.16). Without loss of generality, we take $\underset{\sim}{v}^* = \underset{\sim}{0}$. Let $U = (\underset{\sim}{u} - \underset{\sim}{u}^*, \underset{\sim}{v} - \underset{\sim}{v}^*) \equiv (\underset{\sim}{w}, \underset{\sim}{v})$. Let $\Lambda = (\underset{\sim}{f} - \underset{\sim}{f}^*, \underset{\sim}{q} - \underset{\sim}{q}^*, \underset{\sim}{p} - \underset{\sim}{p}^*, \underset{\sim}{u}_0 - \underset{\sim}{u}^*, \underset{\sim}{u}_1)$ and write $\underset{\sim}{w}_0 = \underset{\sim}{u}_0 - \underset{\sim}{u}^*$, $\underset{\sim}{w}_1 = \underset{\sim}{u}_1$. Then we may write equations (1.1) to (1.4) in the form $F(U,\Lambda) = 0$ with $F = (F_0, F_1, F_2, F_3, F_4)$ where

$$
\begin{aligned}
F_0(U,\Lambda) &= \underset{\sim}{A}[\underset{\sim}{u}]\cdot\ddot{\underset{\sim}{u}} + \underset{\sim}{a}[\underset{\sim}{u}] - \frac{\partial}{\partial s}\underset{\sim}{m}[\underset{\sim}{u}] + \underset{\sim}{n}[\underset{\sim}{u}] - \underset{\sim}{f}[\underset{\sim}{u}] \ , \\
F_1(U,\Lambda) &= \underset{\sim}{u}|_{s_\alpha} - \underset{\sim}{g}[\underset{\sim}{v}] \ , \\
F_2(U,\Lambda) &= (\underset{\sim}{m}[\underset{\sim}{u}]|_{s_\alpha} - \underset{\sim}{p}[\underset{\sim}{u}])\cdot\frac{\partial \underset{\sim}{q}}{\partial \underset{\sim}{v}}[\underset{\sim}{v}] \ , \qquad (4.1) \\
F_3(U,\Lambda) &= \underset{\sim}{w}(\cdot,0) - \underset{\sim}{w}_0 \ , \\
F_4(U,\Lambda) &= \underset{\sim}{w}_t(\cdot,0) - \underset{\sim}{w}_1 \ .
\end{aligned}
$$

To apply the results of Sections 2 and 3, we consider

$$
\begin{aligned}
&\underset{\sim}{u} - \underset{\sim}{u}^* \in Y^\alpha_\sigma(Q_\infty) \ , \qquad (4.2) \\
&\underset{\sim}{v} \in X^{2+\frac{1}{2}\alpha}(\{s_\alpha\} \times \mathbb{R}^+; \mathbb{R}^N \oplus \mathbb{R}^N) \ .
\end{aligned}
$$

If $\underset{\sim}{u}^* \in X^{2+\alpha}$ then there exists an open set $\mathcal{D}$ of values of $\underset{\sim}{u}, \underset{\sim}{v}$, derivatives of $\underset{\sim}{u}$ appearing in (4.1), s, and t, existing in the appropriate product of $\mathbb{R}$ and Q_∞, containing the Q_∞ axis, and such that $F(U,0)$ is defined when the graph of $\underset{\sim}{u}$, $\underset{\sim}{v}$, and the derivatives of $\underset{\sim}{u}$ is in $\mathcal{D}$. (To be precise, we should list seven distinct sets $\mathcal{D}$, we prefer to be slightly ambiguous.) Making $\mathcal{D}$ smaller if necessary we may satisfy the starshaped hypothesis of Proposition 2.14. We take Λ in the following product space:

$$(4.3)\qquad \begin{aligned} &\underset{\sim}{f} - \underset{\sim}{f}^* \in A_\sigma^{1,\alpha,\frac{1}{2}\alpha}(D;\mathbb{R}^N)\,,\\ &\underset{\sim}{q} - \underset{\sim}{q}^* \in A_\sigma^{2,2+\frac{1}{2}\alpha}(D;\mathbb{R}^N \oplus \mathbb{R}^N)\,,\\ &\underset{\sim}{p} - \underset{\sim}{p}^* \in A_\sigma^{1,\frac{1}{2}\alpha}(D;\mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2})\,,\\ &\underset{\sim}{w}_0, \underset{\sim}{w}_1 \in X^{2+\alpha}(s_1,s_2;\mathbb{R}^N)\ . \end{aligned}$$

We denote by P the set of Λ satisfying (4.3) and the following compatibility condition:

(4.4) There exist $U \in Y_\sigma^\alpha(Q_\infty) \times X^{2+\frac{1}{2}\alpha}$ such that $F(U,\Lambda)(s_\alpha,0) = 0$.

We require the following regularity conditions on the remaining functions in equation (1.1):

$$(4.5)\qquad \begin{aligned} &\underset{\sim}{A} \in A^{1,\alpha}(D;\ L(\mathbb{R}^N;\ \mathbb{R}^N))\,,\\ &\underset{\sim}{a} \in A^{1,\alpha}(D;\mathbb{R}^N)\,,\\ &\underset{\sim}{m} \in A^{2,1+\alpha}(D;\mathbb{R}^N)\,,\\ &\underset{\sim}{n} \in A^{1,\alpha}(D;\mathbb{R}^N)\ . \end{aligned}$$

Let D be the Banach space of functions $(\underset{\sim}{f},\underset{\sim}{q},\underset{\sim}{p},\underset{\sim}{w}_0,\underset{\sim}{w}_1)$ satisfying conditions (3.10) with $\mathbb{C}^N$ replaced by $\mathbb{R}^N$.

To state Lemma 4.6 we introduce the following notation: For any function $\underset{\sim}{f}$ of $(\underset{\sim}{u}_s,\underset{\sim}{u},\underset{\sim}{u}_{s_t},\underset{\sim}{u}_t,s,t)$ or subset of these and for any functions $\underset{\sim}{u}$, $\underset{\sim}{w}$ of (s,t) we let

$$\frac{\partial \tilde{f}}{\partial[\underset{\sim}{u}]}[\underset{\sim}{u}]\cdot[\underset{\sim}{w}] = \frac{\partial \tilde{f}}{\partial \underset{\sim}{u}_s}[\underset{\sim}{u}]\cdot\underset{\sim}{w}_s + \frac{\partial \tilde{f}}{\partial \underset{\sim}{u}}[\underset{\sim}{u}]\cdot\underset{\sim}{w} + \frac{\partial \tilde{f}}{\partial \underset{\sim}{u}_{s_t}}[\underset{\sim}{u}]\cdot\underset{\sim}{w}_{s_t} + \frac{\partial \tilde{f}}{\partial \underset{\sim}{u}_t}[\underset{\sim}{u}]\cdot\underset{\sim}{w}_t\ .$$

<u>4.6. Lemma.</u> <u>Let</u> D,P <u>and</u> D <u>be as above and let the regularity conditions</u> (4.5) <u>hold:</u> <u>Then</u> F <u>maps</u> $Y_\sigma^\alpha(Q_\infty) \times X_\sigma^{2+\alpha}(Q_\infty;\mathbb{R}^N + \mathbb{R}^N) \times P$ <u>into</u> D <u>and is continuously differentiable in</u> U. <u>The derivative at</u> $U = 0$, $\Lambda = 0$ <u>is given by</u>

$$
\begin{aligned}
&\left[\frac{\partial F_0}{\partial U}(0,0)\right](\underset{\sim}{w},\underset{\sim}{v}) = P\underset{\sim}{w} - R\dot{\underset{\sim}{w}}_t - S\underset{\sim}{w}\ ,\\
&\left[\frac{\partial F_1}{\partial U}(0,0)\right](\underset{\sim}{w},\underset{\sim}{v}) = \underset{\sim}{w}|_{s_\alpha} - \underset{\sim}{H}\cdot\underset{\sim}{v}\ ,\\
(4.7)\qquad &\left[\frac{\partial F_2}{\partial U}(0,0)\right](\underset{\sim}{w},\underset{\sim}{v}) = B\dot{\underset{\sim}{w}} + \hat{D}\underset{\sim}{w} + (\underset{\sim}{m}[\underset{\sim}{u}^*] - \underset{\sim}{p}^*[\underset{\sim}{u}^*])\cdot\frac{\partial^2\underset{\sim}{q}^*}{\partial \underset{\sim}{v}^2}[\underset{\sim}{0}]\cdot\underset{\sim}{v}\ ,\\
&\left[\frac{\partial F_3}{\partial U}(0,0)\right](\underset{\sim}{w},\underset{\sim}{v}) = \underset{\sim}{w}(\cdot,0)\ ,\\
&\left[\frac{\partial F_4}{\partial U}(0,0)\right](\underset{\sim}{w},\underset{\sim}{v}) = \dot{\underset{\sim}{w}}(\cdot,0)\ ,
\end{aligned}
$$

<u>where</u> $P, R, S, B,$ <u>and</u> $\hat{D}$ <u>have the form given in</u> (3.5) <u>to</u> (3.9). <u>We have</u>

$$
P\underset{\sim}{w}_{tt} - R\underset{\sim}{w}_t - S\underset{\sim}{w} = \underset{\sim}{A}[\underset{\sim}{u}^*]\cdot\underset{\sim}{w}_{tt} - \frac{\partial}{\partial s}\frac{\partial\underset{\sim}{m}}{\partial[\underset{\sim}{u}]}[\underset{\sim}{u}^*]\cdot[\underset{\sim}{w}] + \frac{\partial\underset{\sim}{n}}{\partial[\underset{\sim}{u}]}[\underset{\sim}{u}^*]\cdot[\underset{\sim}{w}] - \frac{\partial\underset{\sim}{f}^*}{\partial[\underset{\sim}{u}]}[\underset{\sim}{u}^*]\cdot[\underset{\sim}{w}]\ ,
$$

$$
(4.8)\qquad B\underset{\sim}{w}_t + \hat{D}\underset{\sim}{w} = \frac{\partial\underset{\sim}{m}}{\partial[\underset{\sim}{u}]}[\underset{\sim}{u}^*]\cdot[\underset{\sim}{w}_s] - \frac{\partial\underset{\sim}{p}^*}{\partial[\underset{\sim}{u}]}[\underset{\sim}{u}^*]\cdot[\underset{\sim}{w}]\cdot\underset{\sim}{H}\ ,
$$

$$
\underset{\sim}{H} = \frac{\partial\underset{\sim}{q}^*}{\partial\underset{\sim}{v}}[\underset{\sim}{0}]\ .
$$

<u>In particular</u>,

$$
\begin{aligned}
&\underset{\sim}{P}_0 = \underset{\sim}{A}[\underset{\sim}{u}^*]\ ,\\
&\underset{\sim}{R}_0 = \frac{\partial\underset{\sim}{m}}{\partial\underset{\sim}{u}_{s_t}}[\underset{\sim}{u}^*]\ ,\\
(4.9)\qquad &\underset{\sim}{S}_0 = \frac{\partial\underset{\sim}{m}}{\partial\underset{\sim}{u}_s}[\underset{\sim}{u}^*]\ ,\\
&\underset{\sim}{B}_0 = \left(\frac{\partial\underset{\sim}{m}}{\partial\underset{\sim}{u}_{s_t}}[\underset{\sim}{u}^*]\,|_{s_\alpha}\right)\cdot\frac{\partial\underset{\sim}{q}}{\partial\underset{\sim}{v}}[\underset{\sim}{0}]\ .
\end{aligned}
$$

Proof. We write out the term $\frac{\partial}{\partial s}\underset{\sim}{m}$ in (4.1) using the chainrule. Under our hypothesis, Proposition 2.17 applies and justifies all formal computations. The derivatives of $\underset{\sim}{A}$ are multiplied by $\underset{\sim}{u}_{tt}$, which is zero when $\underset{\sim}{u} = \underset{\sim}{u}^*$. By (1.9), $\underset{\sim}{a}$ is quadratic in $\underset{\sim}{u}_t$, so that its derivatives drop

out when $\underset{\sim}{u} = \underset{\sim}{u}^*$. The terms resulting from expanding $\frac{\partial \underset{\sim}{m}}{\partial s}$ are regrouped to give (4.8). □

Let D_0 be the subspace of D satisfying the following compatibility condition:

(4.10) If $\Delta \in D_0$ then there exists $U \in Y_0^{\alpha} \times X^{2+\frac{1}{2}\alpha}$ such that

$$\left[\frac{\partial F}{\partial U}(0,0)\right] U(s_\alpha, 0) = \Delta(s_\alpha, 0) .$$

Whether $\Delta \in D$ is an element of D_0 is actually determined by the values of the derivatives of the components of Δ at $(s_\alpha, 0)$. Since D_0 may be characterized as the kernal of a bounded linear operator on D, D_0 is a Banach space.

4.11. Lemma. There exists a function $(\Delta, \Lambda) \in D \times P \mapsto G(\Delta, \Lambda) \in D$, continuously differentiable in D, such that

(i) $G(0,\Lambda) = 0$ for all $\Lambda \in P$,

(ii) $\frac{\partial G}{\partial \Delta}(0,0)$ is the identity on D ,

(iii) If $\Delta = F(U,\Lambda)$ for some U then $G(\Delta,\Lambda) \in D_0$.

Proof. Given $(\Delta,\Lambda) \in D \times P$, we construct $\hat{U}(\Delta,\Lambda)$ such that if $\Delta = F(U,\Lambda)$ for some U then $\Delta - F(\hat{U}(\Delta,\Lambda)\Lambda)(s_\alpha,0) = 0$. Let $\underset{\sim}{H}^{-1} \in L(\mathbb{R}^N \oplus \mathbb{R}^N; \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2})$ be a left inverse to $\underset{\sim}{H}$. Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth function, $\phi(t) = 1$ for $t < \frac{1}{2}$ and $\phi(t) = 0$ for $t > 1$. Using the definitions of F_0, F_3, and F_4, we solve the equation $F(U,\Lambda) = \Delta$ for $\underset{\sim}{w}_0$, $\underset{\sim}{w}_1$, $\underset{\sim}{w}_2$, the initial value and first two initial derivatives of $\underset{\sim}{w}$. We then take

$$\underset{\sim}{w}(\cdot,t) = \phi(t)(\underset{\sim}{w}_0 + t\underset{\sim}{w}_1 + \tfrac{1}{2}t^2\underset{\sim}{w}_2)(\cdot,t)$$

$$\underset{\sim}{v} = \underset{\sim}{H}^{-1} \cdot \underset{\sim}{w}_s ,$$

$$\hat{U}(\Delta,\Lambda) = (\underset{\sim}{w},\underset{\sim}{v}) .$$

If $F(U,\Lambda) = \Delta$ the initial values and first two initial derivatives of U and $\hat{U}(\Delta,\Lambda)$ agree; thus $\Delta - F(\hat{U}(\Delta,\Lambda),\Lambda)(s_\alpha,0) = 0$. By Proposition 2.17,

$\hat{U}$ is a continuously differentiable function of Δ. Note that $\hat{U}(0,0) = 0$.

Let

$$G_0(\Delta,\Lambda) = \Delta - F(\hat{U}(\Delta,\Lambda),\Lambda) + \left[\frac{\partial F}{\partial U}(0,0)\right]\hat{U}(\Delta,\Lambda) .$$

If $\Delta = F(U,\Lambda)$ for some U then $G_0(\Delta,\Lambda) \in D_0$. In fact, if $(\Delta - F(U,\Lambda))(s_\alpha,0) = 0$ for some U then $G_0(\Delta,\Lambda) \in D_0$. By definition of P, (4.4), $G_0(0,\Lambda) \in D_0$. Thus we may take

$$G(\Delta,\Lambda) = G_0(\Delta,\Lambda) - G_0(0,\Lambda) .$$

Conditions (i) and (ii) hold by construction, G is continuously differentiable in Δ by Lemma 4.6 and the composition of differentiable functions, and condition (iii) is readily verified. $\square$

4.12. Theorem. Let $\tilde{Z}$ be the operator defined by (3.38) and suppose $\tilde{Z}$ is analytic for $\mathrm{Re}(\zeta) > -\sigma_0$. Then if $0 \leq \sigma < \sigma_0$ there exists a neighborhood $\mathcal{O}$ of 0 in P such that the equation $F(U,\Lambda) = 0$ has a solution U for all $\Lambda \in \mathcal{O}$, and U depends continuously on Λ.

Thus if $U = (\underset{\sim}{w},\underset{\sim}{v})$ then $(\underset{\sim}{u},\underset{\sim}{v}) = (\underset{\sim}{u}^* + \underset{\sim}{w},\underset{\sim}{v})$ is a solution to equations (1.1) to (1.4) with data related to the stationary data by (4.3). All derivatives occurring in the differential equations are Hölder continuous and the solution decays to $\underset{\sim}{u}^*$ at an exponential rate $e^{-\sigma t}$.

Proof. We apply the implicit function theorem to the equation

$$G(F(U,\Lambda),\Lambda) = 0 , \tag{4.13}$$

where G is the function given in Lemma 4.11. By Lemmas 4.6 and 4.11, the function in equation (4.14) is continuously differentiable in U and maps $Y_0^\alpha \times X^{2+\alpha} \times P_\sigma$ into D_0. By part (i) of Lemma 4.11, equation (4.14) is equivalent to the equation $F(U,\Lambda) = 0$. By part (iii) of Lemma 4.11, we need only show that $\frac{\partial F}{\partial U}(0,0)$ has a bounded inverse. If

$$\left[\frac{\partial F}{\partial U}(0,0)\right](\underset{\sim}{w},\underset{\sim}{v}) = \Delta \tag{4.14}$$

then $\underset{\sim}{v} = \underset{\sim}{H}^{-1}\cdot\underset{\sim}{w}_s$ where $\underset{\sim}{H}^{-1}$ is as in Lemma 4.11. We eliminate $\underset{\sim}{v}$ from

equation (4.15), obtaining equations (3.1) to (3.4) with

$$(4.15) \qquad \underset{\sim}{D}\underset{\sim}{w} = \hat{\underset{\sim}{D}}\underset{\sim}{w} + (\underset{\sim}{m}[\underset{\sim}{u}^*] - \underset{\sim}{p}^*[\underset{\sim}{u}^*]) \cdot \frac{\partial^2 \underset{\sim}{q}}{\partial \underset{\sim}{v}}[0] \cdot \underset{\sim}{H}^{-1} \cdot \underset{\sim}{w}|_{s_\alpha} .$$

We observe that the spaces D_0 and F_σ are equal. By conditions (1.6), (1.7), (1.8) and by (4.9), the conditions on $\underset{\sim}{P}_0$, $\underset{\sim}{R}_0$, and $\underset{\sim}{B}_0$ in Section 3 are met. Then equation (4.15) has a solution and estimate (3.12) holds. But $|\underset{\sim}{v}|_{2+\frac{1}{2}\alpha} < \|\underset{\sim}{H}^{-1}\| \cdot |\underset{\sim}{w}|_{Y_\sigma}$. Therefore $\frac{\partial F}{\partial U}(0,0)$ has a bounded inverse. The continuous dependence of U on Λ is part of the implicit function theorem. □

<u>4.16.</u> <u>Theorem.</u> <u>Let the functions</u> $\underset{\sim}{A}$, $\underset{\sim}{a}$, $\underset{\sim}{m}$, $\underset{\sim}{n}$, $\underset{\sim}{f}$, $\underset{\sim}{q}$, $\underset{\sim}{p}$, $\underset{\sim}{u}_0$, <u>and</u> $\underset{\sim}{u}_1$ <u>have the regularity specified in</u> (4.3) <u>and</u> (4.5), <u>and suppose that</u> $\underset{\sim}{u} \in Y^\alpha(\Omega_T)$ <u>and</u> $\underset{\sim}{v} \in X^{2+\frac{1}{2}\alpha}(0,T)$ <u>are a solution of equations</u> (1.1) <u>to</u> (1.4). <u>Then</u> $\underset{\sim}{u}$ <u>and</u> $\underset{\sim}{v}$ <u>are unique in</u> Y_σ <u>and</u> $X^{2+\frac{1}{2}\alpha}$.

<u>Proof.</u> Let $\underset{\sim}{u}' \in Y^\alpha(\Omega_T)$ and $\underset{\sim}{v}' \in X^{2+\frac{1}{2}\alpha}(0,T)$ be solutions. We shall show that, for some $T' > 0$, $\underset{\sim}{u} = \underset{\sim}{u}'$ and $\underset{\sim}{v} = \underset{\sim}{v}'$ on $0 \leq t \leq T'$. The set of T' having this property is then none-empty and, by repitition of the argument, open, but the set is clearly closed in $(0,T]$, thus is the entire interval.

We may linearize equations (1.1) to (1.4) about the solution $(\underset{\sim}{u},\underset{\sim}{v})$ and apply the implicit function theorem to show that if $(\underset{\sim}{u}',\underset{\sim}{v}')$ is another solution in $Y^\alpha(\Omega_{T'}) \times C^{2+\frac{1}{2}\alpha}(0,T')$ then $|\underset{\sim}{u} - \underset{\sim}{u}'|_{Y^\alpha} + |\underset{\sim}{v} - \underset{\sim}{v}'|_{2+\frac{1}{2}\alpha} > K(T')$. (For finite T our argument does not depend on the constant coefficients in equations (3.1) or (3.4).) We may arrange that K is a non-decreasing function of T': Recall that in the proof of the implicit function theorem (DIEUDONNÉ, 1960) one examines the function $\left(\frac{\partial F}{\partial U}(0,0)\right)^{-1}\left(\frac{\partial F}{\partial U}(0,0)U - F(U,\Lambda)\right)$.

If K is so small that when $\sup\{\|U\|,\|\Lambda\|\} < K$ then

$$(4.17) \qquad \left\|\frac{\partial F}{\partial U}(0,0)U - F(U,\Lambda)\right\| < \frac{1}{2}\left\|\left(\frac{\partial F}{\partial U}(0,0)\right)^{-1}\right\|^{-1}$$

and if Λ is further restricted so that $\left(\frac{\partial F}{\partial U}(0,0)\right)^{-1} F(0,\Lambda) < \frac{1}{2}K$ then this function is a contraction taking the ball of radius K into itself; solutions of $F(U,\Lambda) = 0$ for given Λ exist in this ball and are unique. In our application, for fixed K the left hand of inequality (4.17) is a non-decreasing function of T' while the right hand side is a non-increasing function. Thus K may be chosen a non-decreasing function of T'.

Since $\underset{\sim}{u}$ and $\underset{\sim}{u}'$ agree when $t = 0$, $\sup_{Q_{T'}}|\underset{\sim}{u} - \underset{\sim}{u}'| < 2T'^{\alpha}$. Using Proposition 2.1, one then shows that $\lim_{T'\to 0} |\underset{\sim}{u} - \underset{\sim}{u}'|_{C^{\alpha,\frac{1}{2}\alpha}(0,T')} = 0$. Treating the derivatives in the same way, $\lim_{T'\to 0} |\underset{\sim}{u} - \underset{\sim}{u}'|_{Y^{\alpha,\frac{1}{2}\alpha}(Q_{T'})} = 0$. Then, taking T' small enough, we may conclude that $\underset{\sim}{u} = \underset{\sim}{u}'$ on $Q_{T'}$. Likewise, $\underset{\sim}{v} = \underset{\sim}{v}'$ on $[0,T']$. □

5.0. Conservative Problems.

In this section we suppose that the stationary problem (1.10) to (1.12) is conservative as defined by equations (1.13) to (1.15). In this case we shall show that the eigenfunction criterion for stability developed in Theorems 4.13 and 3.49 is equivalent to the second variation test for stability. Thus the stability or instability of $\underset{\sim}{u}^*$ is determined by the elastic part of the linearized equation, no knowledge of the dissipative mechanism beyond the validity of hypothesis (1.7) and (5.2), below, is needed. We remark that only the equilibrium problem need be conservative; our results hold under small non-conservative perturbations.

Let $E(\underset{\sim}{u},\underset{\sim}{v})$ be the energy defined by equation (1.16). We consider E as a functional on $X^{2+\alpha} \times \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}$. By Proposition 2.17, if the regularity assumptions (4.3) and (4.5) hold then E is twice continuously Fréchet differentiable. Let $\tau \to (\underset{\sim}{u}(\tau),\underset{\sim}{v}(\tau))$ be a path on which the boundary condition (1.11) is satisfied and such that $\underset{\sim}{u}(0) = \underset{\sim}{u}^*$, $\underset{\sim}{v}(0) = \underset{\sim}{v}^*$. Then, using equations (1.10), (1.12), (1.13) to (1.15), (4.8) and (4.15), we have

$$(5.1) \qquad \frac{d^2}{d\tau^2} E\bigg|_{\tau=0} = \left(\int_{s_1}^{s_2} (B\underset{\sim}{u}_\tau)\cdot\underset{\sim}{u}_\tau ds + (D\underset{\sim}{u}_\tau)\cdot\underset{\sim}{v}_\tau \bigg|_{s_1}^{s_2} \right)\bigg|_{\tau=0}$$

$$\equiv (\langle \underset{\sim}{u}_\tau, B\underset{\sim}{u}_\tau\rangle + \langle \underset{\sim}{u}_\tau, D\underset{\sim}{u}_\tau\rangle)_{\tau=0} .$$

Being the value of a continuous second derivative, the quadratic form in (5.1) is symmetric.

We require

$$(5.2) \qquad \langle \underset{\sim}{w}, R\underset{\sim}{w}\rangle + \langle \underset{\sim}{w}, S\underset{\sim}{w}\rangle > 0 \quad \text{for all } \underset{\sim}{w} \neq \underset{\sim}{0} \text{ such that } \underset{\sim}{w}\big|_{s_\alpha} \in \text{Range } \underset{\sim}{H} ,$$

where the quadratic form in (5.2) is defined by analogy with (5.1). Conditions ensuring hypothesis (5.2) are given in BROWNE (1976). Essentially, (5.2) is valid for bodies in which every non-rigid motion suffers internal

friction if the position boundary conditions (1.11) do not admit a deformation $\underset{\sim}{u}$ having the interpretation of a rigid motion.

5.3. Theorem. *Suppose*

(5.4) $\quad \langle \underset{\sim}{w}, S\underset{\sim}{w}\rangle + \langle \underset{\sim}{w}, D\underset{\sim}{w}\rangle > 0$ *for all* $\underset{\sim}{w} \neq \underset{\sim}{0}$ *such that* $\underset{\sim}{w}|_{S_\alpha} \in \text{Range } \underset{\sim}{H}$,

and that hypothesis (5.2) *holds. Then* $\tilde{Z}$ *is analytic for* $\text{Re}(\zeta) \geq 0$.

Thus we may take σ_0 in Theorem 3.47 to be positive, and there exists a σ for which Theorem 4.12 holds.

Proof. By Theorem 3.50, $\tilde{Z}$ is meromorphic for $\text{Re}(\zeta) \geq 0$. Suppose $\tilde{Z}$ has a pole at ζ_0. Then by Theorem 3.50, there exists a non-trivial $\underset{\sim}{\tilde{w}}$ satisfying equation (3.52). The function $\underset{\sim}{w}$ defined by

(5.5) $$\underset{\sim}{w}(s,t) = \text{Re}(e^{\zeta_0 t})\underset{\sim}{\tilde{w}}(s)$$

solves

(5.6) $$\begin{aligned} &P\underset{\sim}{w}_{tt} - R\underset{\sim}{w}_t - S\underset{\sim}{w} = \underset{\sim}{0}, \\ &\underset{\sim}{w}|_{S_\alpha} \in \text{Range } \underset{\sim}{H} , \\ &B\underset{\sim}{w}_t + D\underset{\sim}{w} = \underset{\sim}{0} . \end{aligned}$$

Taking the inner product of the first and last equations in (5.6) with $\underset{\sim}{w}_t$, and using that P and $S + D$ are symmetric, we obtain

$$\frac{1}{2}\frac{d}{dt}[\langle \underset{\sim}{w}_t, P\underset{\sim}{w}_t\rangle + \langle \underset{\sim}{w}, S\underset{\sim}{w}\rangle + \langle \underset{\sim}{w}, D\underset{\sim}{w}\rangle] + \langle \underset{\sim}{w}_t, R\underset{\sim}{w}_t\rangle + \langle \underset{\sim}{w}_t, B\underset{\sim}{w}_t\rangle = 0 .$$

Then, by hypothesis (5.2),

(5.7) $$\frac{1}{2}\frac{d}{dt}[\langle \underset{\sim}{w}_t, P\underset{\sim}{w}_t\rangle + \langle \underset{\sim}{w}, S\underset{\sim}{w}\rangle + \langle \underset{\sim}{w}, D\underset{\sim}{w}\rangle] < 0 \quad \text{for } \underset{\sim}{w}_t \neq \underset{\sim}{0} .$$

But (5.4), (5.5), and (5.7) are contradictory unless $\text{Re}(\zeta_0) < 0$ or $\zeta_0 = 0$. But if $\zeta_0 = 0$ then (5.4) can not hold. □

We finish with a converse to Theorem 5.3.

5.8. Theorem. *Suppose* $\underset{\sim}{u}^* \in X^{2+\alpha}$ *is an isolated solution to the stationary equations* (1.10) *to* (1.12), *that the stationary problem is conservative, that hypothesis* (5.2) *holds, and that there exists* $\underset{\sim}{w} \in X^{2+\alpha}$ *such that*

$$
(5.9) \qquad \begin{aligned} &\langle \underset{\sim}{w}, S\underset{\sim}{w} \rangle + \langle \underset{\sim}{w}, D\underset{\sim}{w} \rangle < 0 \\ &\underset{\sim}{w}\big|_{s_\alpha} \in \text{Range } \underset{\sim}{H} \, . \end{aligned}
$$

Then $\underset{\sim}{u}^*$ is not stable in $X^{2+\alpha}$.

Proof. We shall consider only perturbations in the initial data. Suppose, for the sake of contradiction, that for every open neighborhood $\mathcal{O}_1$ of $(\underset{\sim}{u}^*, \underset{\sim}{0})$ in $X^{2+\alpha} \times X^{2+\alpha}$ there exists an open neighborhood $\mathcal{O}_0$ such that for all $(\underset{\sim}{u}_0, \underset{\sim}{u}_1) \in \mathcal{O}_0$ equations (1.1), (1.4), (1.11), and (1.12) have a solution $\underset{\sim}{u}$, and for all $t \geq 0$ $(\underset{\sim}{u}(t), \underset{\sim}{u}_t(t)) \in \mathcal{O}_1$. Taking the inner product of equation (1.1) with $\underset{\sim}{u}_t$ and integrating by parts gives

$$
(5.10) \qquad \begin{aligned} \frac{d}{dt} E(\underset{\sim}{u}(\cdot,t)) + \int_{s_1}^{s_2} \{ [\underset{\sim}{m}[\underset{\sim}{u}] - \underset{\sim}{m}[\underset{\sim}{u}(\cdot,t)]] \cdot \underset{\sim}{u}_{st} \\ + [\underset{\sim}{n}[\underset{\sim}{u}] - \underset{\sim}{n}[\underset{\sim}{u}(\cdot,t)]] \cdot \underset{\sim}{u}_t \} ds + [\underset{\sim}{m}[\underset{\sim}{u}] - \underset{\sim}{m}[\underset{\sim}{u}(\cdot,t)]] \cdot \underset{\sim}{u}_t \big|_{s_\alpha} \end{aligned}
$$

(where $\underset{\sim}{m}[\underset{\sim}{u}(\cdot,t)]$ represents $\underset{\sim}{m}(\underset{\sim}{u}_s, \underset{\sim}{u}, \underset{\sim}{0}, \underset{\sim}{0}, s))$. If $\mathcal{O}_1$ is suitably small we may estimate the integrand in (5.10) by $\langle \underset{\sim}{u}_t, R\underset{\sim}{u}_t \rangle + \langle \underset{\sim}{u}_t, B\underset{\sim}{u}_t \rangle$, and conclude from (5.2) that

$$
(5.11) \qquad \frac{d}{dt} E(\underset{\sim}{u}(\cdot,t)) < 0 \quad \text{if} \quad \underset{\sim}{u}_t \neq 0 \, .
$$

Now let $0 < \beta < \alpha$, and for any $t \geq 0$ let

$$
\Omega_t = cl_{2+\beta} \{ (\underset{\sim}{u}(\cdot,\tau), u_t(\cdot,\tau)) \; : \; \tau > t \}
$$

where $cl_{2+\beta}$ represents the closure operator in the topology of $C^{2+\beta}$. Since $X^{2+\alpha}$ imbeds compactly in $X^{2+\beta}$, Ω_t is compact, and $\Omega_\infty \equiv \bigcap_{t>0} \Omega_t$ is non-empty. If $\hat{\underset{\sim}{u}}$ is the solution of equations (1.1), (1.4), (1.11), and (1.12) with initial data $(\hat{u}_0, \hat{u}_1) \in \Omega_\infty$ then $(\hat{u}(\cdot,t), \hat{u}_t(\cdot,t)) \in \Omega_\infty$ for all t. Since $E(\hat{\underset{\sim}{u}}_0) = \lim_{t\to\infty} E(\underset{\sim}{u}(\cdot,t))$, E is constant on Ω_∞. Then, by (5.11), $\hat{u}_0$ is a stationary solution, and, taking $\mathcal{O}_1$ small enough, $\hat{\underset{\sim}{u}}_0 = \underset{\sim}{u}^*$. But if (5.9) holds then we may choose $\underset{\sim}{u}_0$ so that, by 5.11,

$$
E(\hat{\underset{\sim}{u}}_0) < E(\underset{\sim}{u}_0) < E(\underset{\sim}{u}^*) \; ;
$$

then $\underset{\sim}{u}_0 \neq \underset{\sim}{u}^*$. $\square$

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